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Two-sided coupled generalized Sylvester matrix equations solving using a simultaneous decomposition for fifteen matrices

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ABSTRACT

In this paper, we investigate and analyze in detail the structure and properties of a simultaneous decomposition for fifteen matrices: $A_i \in \mathbb{C}^{p_i \times t_i}$, $B_i \in \mathbb{C}^{s_i \times q_i}$, $C_i \in \mathbb{C}^{p_i \times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1} \times q_i}$, and $E_i \in \mathbb{C}^{p_i \times q_i}$ $(i = 1, 2, 3)$. We show that from this simultaneous decomposition we can derive some necessary and sufficient conditions for the existence of a solution to the system of two-sided coupled generalized Sylvester matrix equations with four unknowns $A_i X_i B_i + C_i X_{i+1} D_i = E_i$ $(i = 1, 2, 3)$. Apart from proving an expression for the general solutions to this system, we derive the range of ranks of these solutions using the ranks of the given matrices A_i, B_i, C_i, D_i, and E_i. We provide some numerical examples to illustrate our results. Moreover, we present a similar approach to consider the simultaneous decomposition for $5k$ matrices and the system of k two-sided coupled generalized Sylvester matrix equations with $k + 1$ unknowns $A_i X_i B_i + C_i X_{i+1} D_i = E_i$ $(i = 1, \ldots, k,\ k \geq 4)$.

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The main results are also valid over the real number field and the real quaternion algebra.
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1. Introduction

Let $\mathbb{C}$ and $\mathbb{C}^{m\times n}$ stand, respectively, for the complex number field, and the set of all $m \times n$ matrices over $\mathbb{C}$. In this paper, we study the system of two-sided coupled generalized Sylvester matrix equations with four unknowns

$$A_iX_iB_i + C_iX_{i+1}D_i = E_i \quad (i = 1, 2, 3), \tag{1}$$

where $A_i \in \mathbb{C}^{p_i\times t_i}$, $B_i \in \mathbb{C}^{s_i\times q_i}$, $C_i \in \mathbb{C}^{p_i\times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1}\times q_i}$, and $E_i \in \mathbb{C}^{p_i\times q_i}$ are given matrices. We present some solvability conditions, general solutions, as well as the range of ranks of the general solutions to the above-mentioned system (1) through a simultaneous decomposition of the coefficient matrices A_i, B_i, C_i, D_i, and E_i. Moreover, we consider the system of k two-sided coupled generalized Sylvester matrix equations with $k + 1$ unknowns

$$A_iX_iB_i + C_iX_{i+1}D_i = E_i,\ i = 1, 2, \ldots, k,\ k \geq 4. \tag{2}$$

The system (1) contains the existing generalizations of the well-known one-sided generalized (mixed) Sylvester matrix equations

$$A_1X + YD_1 = E_1, \tag{3}$$

$$A_jX_j - YB_j = C_j \quad (j = 1, 2), \tag{4}$$

$$A_jX_j - X_{j+1}B_j = C_j \quad (j = 1, 2), \tag{5}$$

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_2 - X_3B_2 = C_2, \\ A_3X_3 - X_4B_3 = C_3. \end{cases} \quad \begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_3 - X_2B_2 = C_2, \\ A_3X_3 - X_4B_3 = C_3, \end{cases} \quad \begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_2 - X_3B_2 = C_2, \\ A_3X_4 - X_3B_3 = C_3, \end{cases} \tag{6}$$

and the classical matrix equations

$$C_jZD_j = E_j \quad (j = 1, 2), \tag{7}$$

as special cases.

Sylvester-type matrix equations have wide applications in several fields. For example in economics, Sylvester equations are involved in the higher-order perturbation

methods developed for solving Stochastic Dynamic General Equilibrium (SDGE) models [35]. In computer vision, pose estimation from multiple cameras can be obtained as a solution to a generalized Sylvester equation [8]. Furthermore, Sylvester equations arise from image restoration problems like the ones described in [7]. They are also used in the derivation of the numerical solution of implicit ordinary differential equations [25], they appear when dealing with the problem of reordering eigenvalues of regular matrix pairs [33], and they arise when computing the deflating subspace of descriptor linear systems [34]. Sylvester equations (containing the Lyapunov matrix equation as a special case) have been used extensively in the field of systems and control. For instance, these equations have been used in the stability analysis of dynamical systems [27], in robust control [6,45,53], in the design of state-feedback controllers and state observers via pole placement [27], in robust fault detection [9,23,44], in output feedback control [36,48,51], in reduced observer and compensator design problems [51], in neural-network-based control [64], in model reduction [3,28], in the finite transmission zero assignment problem [51,52], in the control of singular systems [24,47], in the almost noninteracting control by measurement feedback problem [58,60], graph theory [22], etc. When $X_1 = X_2$, the system (4) becomes $A_jX_1 - YB_j = C_j$, $(j = 1, 2)$, which arises when solving generalized eigenvalue problems [50,58].

Roth [46] proved that if $\begin{pmatrix} A_1 & E_1 \\ 0 & D_1 \end{pmatrix}$ and $\begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}$ are equivalent, then the one-sided generalized Sylvester matrix equation (3) is consistent. Baksalary and Kala [4] gave an expression of the general solution to (3) by using generalized inverses. Wang et al. [57] presented some necessary and sufficient conditions for the existence of bi(skew)symmetric solutions to some systems of generalized Sylvester equations. Recently, some researchers considered the one-sided generalized mixed Sylvester matrix equations (4) and (5). Lee and Vu [37] presented a solvability condition of the one-sided generalized mixed Sylvester matrix equations (4). Wang and He derived some solvability conditions and general solutions to the systems (4), (5), and (6) using ranks and generalized inverses [29,54,55]. Jonsson and Kågström [31,32] gave some effective approaches for solving one-sided and two-sided triangular Sylvester-type matrix equations. De Terán et al. [21] described how to find the general solution of the $\star$-Sylvester equation $AX + X^*B = 0$. Recently, Dmytryshyn and Kågström [22] derived some solvability conditions of systems consisting of Sylvester and $\star$-Sylvester equations through the corresponding equivalence relations of the block matrices. The Stein-type matrix equation has been studied by many authors (e.g., [37,59,65]). The classical matrix equations (7) have found some applications in the field of almost (or approximate) non interacting control [60]. Since Mitra [40] first considered the classical matrix equations (7) in 1973, there have been many papers discussing this system (e.g. [42,56,60]).

Motivated by the wide application of the generalized Sylvester matrix equation and in order to improve the theoretical development of generalized Sylvester matrix equations, we consider the system of two-sided coupled generalized Sylvester matrix equations (1),

which is a generalization of those in [5,29,37,41,54,55,60,62]. Recall the one-sided coupled generalized Sylvester matrix equations (6) (the important partial case of the main system (1)). The authors in [55] gave the solvability conditions and general solutions to (6) using the Moore–Penrose inverses of the matrices. Notice however, that it is hard to give the solvability conditions and general solutions to (1) using the approach presented in [55].

In order to consider the solvability conditions and general solutions to the system (1), we construct a simultaneous decomposition for the coefficient matrices A_i, B_i, C_i, D_i, and E_i. Observe that the dimensions of the coefficient matrices A_i, B_i, C_i, D_i, and E_i have certain rules. The matrices A_i, C_i, and E_i have the same number of rows, meanwhile B_i, D_i, and E_i have the same number of columns. On the other hand, C_i and A_{i+1} have the same number of columns, meanwhile D_i and B_{i+1} have the same number of rows. Hence, the coefficient matrices A_i, B_i, C_i, D_i, and E_i can be arranged in the following matrix array

$$\begin{array}{c} \\ s_1 \\ p_1 \\ s_2 \\ p_2 \\ s_3 \\ p_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} t_1 & q_1 & t_2 & q_2 & t_3 & q_3 & t_4 \end{array} \\ \begin{pmatrix} & B_1 & & & & & \\ A_1 & E_1 & C_1 & & & & \\ & D_1 & & B_2 & & & \\ & & A_2 & E_2 & C_2 & & \\ & & & D_2 & & B_3 & \\ & & & & A_3 & E_3 & C_3 \\ & & & & & D_3 & \end{pmatrix} \end{array}. \tag{8}$$

One contribution of the paper is to present a simultaneous decomposition for the matrix array (8). We show how to find nonsingular matrices $P_i \in \mathbb{C}^{p_i \times p_i}$, $Q_i \in \mathbb{C}^{q_i \times q_i}$, $T_i \in \mathbb{C}^{t_i \times t_i}$, and $S_i \in \mathbb{C}^{s_i \times s_i}$, such that

$$P_i A_i T_i = \Delta_{2i-1},\ P_i C_i T_{i+1} = \Delta_{2i},\ S_i B_i Q_i = \Omega_{2i-1},\ S_{i+1} D_i Q_i = \Omega_{2i},\ P_i E_i Q_i = \Phi_i,$$

where $\Delta_1, \ldots, \Delta_6, \Omega_1, \ldots, \Omega_6$ are quasi-diagonal matrices with the finest possible subdivision of matrices, and Φ_i have appropriate forms. Through the simultaneous matrix decomposition, we derive a canonical form for the system (1) and then give some solvability conditions and general solutions to system (1).

The remainder of the paper is organized as follows. In Section 2, we construct and analyze in detail the structure and properties of a simultaneous decomposition for the matrix array (8). In §3.1, we give a canonical form for the system (1) through the simultaneous matrix decomposition and then derive some necessary and sufficient conditions for the existence of a solution to the system (1). In §3.2, we present the general solutions to the system (1) when the solvability conditions are satisfied. In

§3.3, we derive the range of ranks of the general solutions to the system (1). In Section 4, we present a similar approach to consider the simultaneous decomposition for $5k$ matrices and the system of k two-sided coupled generalized Sylvester matrix equations (2).

Throughout this paper, the symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of a given matrix A, respectively. We use $r_{a_{11}a_{12}\cdots a_{1,n}|a_{21}a_{22}\cdots a_{2,n}|\cdots|a_{m,1}a_{m,2}\cdots a_{m,n}}$ to represent the rank of a block matrix

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,n} \\ A_{21} & A_{22} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix}. \tag{9}$$

We use lowercase letters in subscripts. If a matrix A_{ij} in the block matrix (9) equals zero, we use "0" instead of a_{ij} in the subscript. For instance, $r_{e_1a_1c_1}$, $r_{e_1|b_1|d_1}$, and $r_{a_1c_10|0c_2-a_2}$ stand for the ranks of the following block matrices

$$(E_1, A_1, C_1), \quad \begin{pmatrix} E_1 \\ B_1 \\ D_1 \end{pmatrix}, \quad \begin{pmatrix} A_1 & C_1 & 0 \\ 0 & C_2 & -A_2 \end{pmatrix},$$

respectively. I denotes an identity matrix with appropriate size.

2. A simultaneous decomposition for the matrix array (8)

This section is organized as follows:

- §2.1. The structure of a simultaneous decomposition for the matrix array (8) is proposed.
- §2.2. The implicit expressions of the dimensions of identity matrices in the structure are given.

2.1. Structure of a simultaneous decomposition for the matrix array (8)

Matrix decompositions or matrix factorizations are a fundamental theme in linear algebra and applied statistics which has both scientific and engineering significance. Among the available matrix decompositions, it is worth to mention the ordinary singular value decomposition (OSVD) (e.g., [1,2,49]), the quotient singular value decomposition (QSVD) (e.g., [16,14,39,43]), the product singular value decomposition (PSVD) (e.g., [11,26,30]), the restricted singular value decomposition (RSVD) (e.g., [18,10,63]).

In order to derive a simultaneous matrix decomposition for the matrix array (8), we need the generalized singular value decomposition for any k matrices $(A_1, A_2, \ldots, A_k)$ with $A_1 \in \mathbb{C}^{n_0\times n_1}, A_2 \in \mathbb{C}^{n_1\times n_2}, \ldots, A_k \in \mathbb{C}^{n_{k-1}\times n_k}$. De Moor, Van Dooren, and Zha

analyzed in detail the structure of the generalizations of the singular value decomposition for any number of matrices [17,19,20]. The following result is the generalized singular value decomposition for any k matrices.

Lemma 2.1 *($Q^{k-1}-SVD$ for any k matrices). (See [17,19,20].) Consider a set of k matrices with compatible dimensions: $A_1 \in \mathbb{C}^{n_0\times n_1}$, $A_2 \in \mathbb{C}^{n_1\times n_2}$, $\ldots$, $A_{k-1} \in \mathbb{C}^{n_{k-2}\times n_{k-1}}$, $A_k \in \mathbb{C}^{n_{k-1}\times n_k}$. Then there exist*

—*unitary matrices $U_1 \in \mathbb{C}^{n_0\times n_0}$, $V_k \in \mathbb{C}^{n_k\times n_k}$;*

—*matrices Λ_j, $j = 1, 2, \ldots, k-1$ of the form*

$$\Lambda_j = \begin{array}{c} \\ r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ r_{j-1}^3 - r_j^3 \\ \cdots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{array} \begin{array}{c} \begin{array}{ccccccc} r_j^1 & r_j^2 & r_j^3 & r_j^4 & \cdots & r_j^j & n_j - r_j \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array},$$

$$r_0 = 0,\ r_j = \sum_{i=1}^{j} r_j^i = r_{a_j};$$

—*a matrix Λ_k of the form*

$$\Lambda_k = \begin{array}{c} \\ r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ r_{k-1}^3 - r_k^3 \\ \cdots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{array} \begin{array}{c} \begin{array}{ccccccc} r_k^1 & r_k^2 & r_k^3 & r_k^4 & \cdots & r_k^k & n_k - r_k \end{array} \\ \begin{pmatrix} \Lambda_k^1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda_k^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \Lambda_k^3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \Lambda_k^k & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}.$$

The $r_k^i \times r_k^i$ matrices Λ_k^i are diagonal with positive diagonal elements.
—Nonsingular matrices $X_j \in \mathbb{C}^{n_j \times n_j}, (j = 1, \ldots, k-1)$, such that

$$U_1^* A_1 X_1 = \Lambda_1, X_1^* A_2 X_2 = \Lambda_2, \cdots = \cdots, X_{i-1}^* A_i X_i = \Lambda_i, \cdots = \cdots, X_{k-1}^* A_k V_k = \Lambda_k.$$

Equipped with Lemma 2.1, we now turn attention to establish the simultaneous decomposition for the matrix array (8) through the following three steps.

1. Give an equivalence canonical form of the set of 6 matrices:

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} A_1 & C_1 & & \\ & A_2 & C_2 & \\ & & A_3 & C_3 \end{pmatrix} \end{array}. \tag{10}$$

Note that $A_1^* \in \mathbb{C}^{t_1 \times p_1}$, $C_1 \in \mathbb{C}^{p_1 \times t_2}$, $A_2^* \in \mathbb{C}^{t_2 \times p_2}$, $C_2 \in \mathbb{C}^{p_2 \times t_3}$, $A_3^* \in \mathbb{C}^{t_3 \times p_3}$, $C_3 \in \mathbb{C}^{p_3 \times t_4}$. Applying $Q^5 - SVD$ (i.e., $k = 6$ in Lemma 2.1) to the six matrices A_i^* and C_i gives

$$U_1^* A_1^* X_1 = \Lambda_1, X_1^* C_1 X_2 = \Lambda_2, X_2^* A_2^* X_3 = \Lambda_3, X_3^* C_2 X_4 = \Lambda_4,$$
$$X_4^* A_3^* X_5 = \Lambda_5, X_5^* C_3 X_6 = \Sigma_6,$$

where U_1, V_6 are unitary matrices, $X_1, X_2, X_3, X_4, X_5, X_6 := V_6 \mathrm{diag}(\Lambda_6^1, \Lambda_6^2, \Lambda_6^3, \Lambda_6^4, \Lambda_6^5, \Lambda_6^6)^{-1}$ are nonsingular matrices, and $\Lambda_1, \ldots, \Lambda_5, \Sigma_6$ are quasi-diagonal matrices. Then, we have

$$\mathrm{diag}(X_1^*, X_3^*, X_5^*) \begin{pmatrix} A_1 & C_1 & & \\ & A_2 & C_2 & \\ & & A_3 & C_3 \end{pmatrix} \mathrm{diag}(U_1, X_2, X_4, X_6)$$
$$= \begin{pmatrix} \Lambda_1^* & \Lambda_2 & & \\ & \Lambda_3^* & \Lambda_4 & \\ & & \Lambda_5^* & \Sigma_6 \end{pmatrix}.$$

Hence, we can obtain expressions for all dimensions involved by analyzing the block bidiagonal matrix

$$\begin{pmatrix} \Lambda_1^* & \Lambda_2 & & \\ & \Lambda_3^* & \Lambda_4 & \\ & & \Lambda_5^* & \Sigma_6 \end{pmatrix},$$

where we have used the finest possible subdivision of matrices.

2. Using the similar approach of Step 1, we can give an equivalence canonical form of the set of 6 matrices:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} B_1 & & \\ D_1 & B_2 & \\ & D_2 & B_3 \\ & & D_3 \end{pmatrix} \end{array}. \tag{11}$$

We can find nonsingular matrices $Y_1, \ldots, Y_6$, and a unitary matrix U_2 such that

$$\operatorname{diag}(U_2^*, Y_2^*, Y_4^*, Y_6^*) \begin{pmatrix} B_1 & & \\ D_1 & B_2 & \\ & D_2 & B_3 \\ & & D_3 \end{pmatrix} \operatorname{diag}(Y_1, Y_3, Y_5) = \begin{pmatrix} \Theta_1 & & \\ \Theta_2^* & \Theta_3 & \\ & \Theta_4^* & \Theta_5 \\ & & \Theta_6^* \end{pmatrix},$$

where $\Theta_1, \ldots, \Theta_6$ are quasi-diagonal matrices with the finest possible subdivision of matrices.

3. Applying elementary matrix operations and the identity matrices in the equivalence canonical forms of matrices (10) and (11), we can obtain a simultaneous decomposition of the matrix array (8).

Now we give the main theorem of this section.

Theorem 2.2. *Consider a set of 15 matrices:* $A_i \in \mathbb{C}^{p_i \times t_i}, B_i \in \mathbb{C}^{s_i \times q_i}, C_i \in \mathbb{C}^{p_i \times t_{i+1}}, D_i \in \mathbb{C}^{s_{i+1} \times q_i}$, *and* $E_i \in \mathbb{C}^{p_i \times q_i}$, $(i = 1, 2, 3)$. *Then there exist*

—*Unitary matrices* $T_1 \in \mathbb{C}^{t_1 \times t_1}$ *and* $S_1 \in \mathbb{C}^{s_1 \times s_1}$.

—*Matrices* $\Delta_{2i-1}, \Delta_{2i}, \Omega_{2i-1}, \Omega_{2i}$ *of the form*

$$\begin{pmatrix} \Delta_1 & \Delta_2 & & \\ & \Delta_3 & \Delta_4 & \\ & & \Delta_5 & \Delta_6 \end{pmatrix} = \Psi_1, \begin{pmatrix} \Omega_1 & & & \\ \Omega_2 & \Omega_3 & & \\ & \Omega_4 & \Omega_5 & \\ & & & \Omega_6 \end{pmatrix} = \Psi_2^*, \tag{12}$$

$$\Psi_j = \begin{array}{c} \\ n_{j;1} \\ n_{j;2} \\ n_{j;3} \\ n_{j;4} \\ n_{j;5} \\ n_{j;6} \\ n_{j;7} \\ n_{j;8} \\ n_{j;9} \\ n_{j;10} \\ n_{j;11} \\ \\ l_1 \\ n_{j;1} \\ n_{j;2} \\ n_{j;3} \\ n_{j;4} \\ n_{j;7} \\ n_{j;8} \\ n_{j;9} \\ n_{j;10} \\ n_{j;12} \\ n_{j;13} \\ n_{j;14} \\ n_{j;15} \\ n_{j;16} \\ n_{j;17} \\ n_{j;18} \\ \\ l_2 \\ n_{j;1} \\ n_{j;2} \\ n_{j;7} \\ n_{j;8} \\ n_{j;12} \\ n_{j;13} \\ n_{j;16} \\ n_{j;17} \\ n_{j;19} \\ n_{j;20} \\ n_{j;21} \\ \\ l_3 \end{array} \begin{pmatrix} I_{7\times 7} & I_{7\times 15} & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & I_{15\times 15} & I_{15\times 15} & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & &

$$\Phi_2 = \begin{pmatrix} E_{11}^2 & \cdots & E_{1,15}^2 & E_{1,16}^2 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ E_{15,1}^2 & \cdots & E_{15,15}^2 & E_{15,16}^2 & 0 \\ E_{16,1}^2 & \cdots & E_{16,15}^2 & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_{l_2} \end{pmatrix}. \tag{14}$$

The block rows of Φ_1, Φ_2, and Φ_3 are the same as the block rows of Δ_1, Δ_3, and Δ_5, respectively. The block columns of Φ_1, Φ_2, and Φ_3 are the same as the block rows of Ω_1^*, Ω_3^*, and Ω_5^*, respectively.

— *Nonsingular matrices $P_i \in \mathbb{C}^{p_i \times p_i}, Q_i \in \mathbb{C}^{q_i \times q_i}, T_2 \in \mathbb{C}^{t_2 \times t_2}, T_3 \in \mathbb{C}^{t_3 \times t_3}, T_4 \in \mathbb{C}^{t_4 \times t_4}, S_2 \in \mathbb{C}^{s_2 \times s_2}, S_3 \in \mathbb{C}^{s_3 \times s_3}$, and $S_4 \in \mathbb{C}^{s_4 \times s_4}$, such that*

$$P_i A_i T_i = \Delta_{2i-1},\ P_i C_i T_{i+1} = \Delta_{2i},\ S_i B_i Q_i = \Omega_{2i-1},\ S_{i+1} D_i Q_i = \Omega_{2i},\ P_i E_i Q_i = \Phi_i. \tag{15}$$

Expressions for the integers $n_{j;1}, \ldots, n_{j;21}, l_i$ are given in Theorem 2.4.

Proof. The proof is given in Appendix A. □

Remark 2.3. Observe that the matrices $\Delta_{2i-1}, \Delta_{2i}, \Omega_{2i-1}, \Omega_{2i}$ in (13) are quasi-diagonal matrices with the finest possible subdivision of matrices, their only nonzero blocks being identity matrices. From (15) in Theorem 2.2, we can rewrite the system (1) in the following canonical form

$$\begin{cases} P_1^{-1}\Delta_1 T_1^{-1} X_1 S_1^{-1}\Omega_1 Q_1 + P_1^{-1}\Delta_2 T_2^{-1} X_2 S_2^{-1}\Omega_2 Q_1^{-1} = P_1^{-1}\Phi_1 Q_1^{-1}, \\ P_2^{-1}\Delta_3 T_2^{-1} X_2 S_2^{-1}\Omega_3 Q_2 + P_2^{-1}\Delta_4 T_3^{-1} X_3 S_3^{-1}\Omega_4 Q_2^{-1} = P_2^{-1}\Phi_2 Q_2^{-1}, \\ P_3^{-1}\Delta_5 T_3^{-1} X_3 S_3^{-1}\Omega_5 Q_3 + P_3^{-1}\Delta_6 T_4^{-1} X_4 S_4^{-1}\Omega_6 Q_3^{-1} = P_3^{-1}\Phi_3 Q_3^{-1}. \end{cases}$$

If we then let $Y_k = T_k^{-1} X_k S_k^{-1}$, $(k = 1, 2, 3, 4)$, the original system (1) transforms to

$$\begin{cases} \Delta_1 Y_1 \Omega_1 + \Delta_2 Y_2 \Omega_2 = \Phi_1, \\ \Delta_3 Y_2 \Omega_3 + \Delta_4 Y_3 \Omega_4 = \Phi_2, \\ \Delta_5 Y_3 \Omega_5 + \Delta_6 Y_4 \Omega_6 = \Phi_3. \end{cases} \tag{16}$$

This is still a system of two-sided coupled generalized Sylvester matrix equations with four unknowns Y_k, but now the matrices $\Delta_1, \ldots, \Delta_6, \Omega_1, \ldots, \Omega_6$ are quasi-diagonal matrices, and Φ_i have appropriate forms. We use (16) to give some solvability conditions and general solutions to the system (1).

2.2. Properties of the decomposition for the matrix array (8)

In this section, we derive the direct expressions for the block dimensions $n_{j;1}, \ldots, n_{j;21}, l_i$ in Theorem 2.2 in terms of the ranks of the matrices A_i, B_i, C_i, D_i, and E_i.

Theorem 2.4. *Consider the above factorization for the 15 matrices (8): $A_i \in \mathbb{C}^{p_i\times t_i}, B_i \in \mathbb{C}^{s_i\times q_i}, C_i \in \mathbb{C}^{p_i\times t_{i+1}}, D_i \in \mathbb{C}^{s_{i+1}\times q_i}$, and $E_i \in \mathbb{C}^{p_i\times q_i}$, $(i = 1, 2, 3)$. The block dimensions $n_{j;1}, \ldots, n_{j;21}, l_i$ that appear in Theorem 2.2 are given by:*

$n_{1;1}$	$\begin{matrix} r_{a_1c_10\|0a_2c_2\|00a_3} + r_{c_100\|a_2c_20\|0a_3c_3} \\ -r_{c_10\|a_2c_2\|0a_3} - r_{a_1c_100\|0a_2c_20\|00a_3c_3}\end{matrix}$	$n_{2;1}$	$\begin{matrix} r_{b_100\|d_1b_20\|0d_2b_3} + r_{d_1b_20\|0d_2b_3\|00d_3} \\ -r_{d_1b_20\|0d_2b_3} - r_{b_100\|d_1b_20\|0d_2b_3\|00d_3}\end{matrix}$
$n_{1;2}$	$\begin{matrix} r_{c_10\|a_2c_2} + r_{a_1c_100\|0a_2c_20\|00a_3c_3} \\ -r_{a_1c_10\|0a_2c_2} - r_{c_100\|a_2c_20\|0a_3c_3}\end{matrix}$	$n_{2;2}$	$\begin{matrix} r_{d_1b_2\|0d_2} + r_{b_100\|d_1b_20\|0d_2b_3\|00d_3} \\ -r_{b_10\|d_1b_2\|0d_2} - r_{d_1b_20\|0d_2b_3\|00d_3}\end{matrix}$
$n_{1;3}$	$\begin{matrix} r_{a_1c_1\|0a_2} + r_{c_10\|a_2c_2\|0a_3} \\ -r_{c_1\|a_2} - r_{a_1c_10\|0a_2c_2\|00a_3}\end{matrix}$	$n_{2;3}$	$\begin{matrix} r_{b_10\|d_1b_2} + r_{d_1b_20\|0d_2b_3} \\ -r_{d_1b_2} - r_{b_100\|d_1b_20\|0d_2b_3}\end{matrix}$
$n_{1;4}$	$r_{c_1} + r_{a_1c_10\|0a_2c_2} - r_{a_1c_1} - r_{c_10\|a_2c_2}$	$n_{2;4}$	$r_{d_1} + r_{b_10\|d_1b_2\|0d_2} - r_{b_1\|d_1} - r_{d_1b_2\|0d_2}$
$n_{1;5}$	$r_{a_1} + r_{c_1\|a_2} - r_{a_1c_1\|0a_2}$	$n_{2;5}$	$r_{b_1} + r_{d_1b_2} - r_{b_10\|d_1b_2}$
$n_{1;6}$	$r_{a_1c_1} - r_{c_1}$	$n_{2;6}$	$r_{b_1\|d_1} - r_{d_1}$
$n_{1;7}$	$\begin{matrix} r_{a_2c_2\|0a_3} + r_{a_1c_100\|0a_2c_20\|00a_3c_3} \\ -r_{a_2c_20\|0a_3c_3} - r_{a_1c_10\|0a_2c_2\|00a_3}\end{matrix}$	$n_{2;7}$	$\begin{matrix} r_{b_20\|d_2b_3} + r_{b_100\|d_1b_20\|0d_2b_3\|00d_3} \\ -r_{b_20\|d_2b_3\|0d_3} - r_{b_100\|d_1b_20\|0d_2b_3}\end{matrix}$
$n_{1;8}$	$\begin{matrix} r_{a_1c_10\|0a_2c_2} + r_{a_2c_20\|0a_3c_3} \\ -r_{a_2c_2} - r_{a_1c_100\|0a_2c_20\|00a_3c_3}\end{matrix}$	$n_{2;8}$	$\begin{matrix} r_{b_10\|d_1b_2\|0d_2} + r_{b_20\|d_2b_3\|0d_3} \\ -r_{b_2\|d_2} - r_{b_100\|d_1b_20\|0d_2b_3\|00d_3}\end{matrix}$
$n_{1;9}$	$\begin{matrix} r_{a_2} + r_{a_1c_10\|0a_2c_2\|00a_3} \\ -r_{a_1c_1\|0a_2} - r_{a_2c_2\|0a_3}\end{matrix}$	$n_{2;9}$	$\begin{matrix} r_{b_2} + r_{b_100\|d_1b_20\|0d_2b_3} \\ -r_{b_10\|d_1b_2} - r_{b_20\|d_2b_3}\end{matrix}$
$n_{1;10}$	$r_{a_1c_1} + r_{a_2c_2} - r_{a_1c_10\|0a_2c_2}$	$n_{2;10}$	$r_{b_1\|d_1} + r_{b_2\|d_2} - r_{b_10\|d_1b_2\|0d_2}$
$n_{1;11}$	$r_{a_1c_1\|0a_2} - r_{a_1} - r_{a_2}$	$n_{2;11}$	$r_{b_10\|d_1b_2} - r_{b_1} - r_{b_2}$
$n_{1;12}$	$\begin{matrix} r_{c_20\|a_3c_3} + r_{c_10\|a_2c_2\|0a_3} \\ -r_{c_2\|a_3} - r_{c_100\|a_2c_20\|0a_3c_3}\end{matrix}$	$n_{2;12}$	$\begin{matrix} r_{d_2b_3\|0d_3} + r_{d_1b_20\|0d_2b_3} \\ -r_{d_2b_3} - r_{d_1b_20\|0d_2b_3\|00d_3}\end{matrix}$
$n_{1;13}$	$\begin{matrix} r_{c_2} + r_{c_100\|a_2c_20\|0a_3c_3} \\ -r_{c_10\|a_2c_2} - r_{c_20\|a_3c_3}\end{matrix}$	$n_{2;13}$	$\begin{matrix} r_{d_2} + r_{d_1b_20\|0d_2b_3\|00d_3} \\ -r_{d_1b_2\|0d_2} - r_{d_2b_3\|0d_3}\end{matrix}$
$n_{1;14}$	$r_{c_1\|a_2} + r_{c_2\|a_3} - r_{c_10\|a_2c_2\|0a_3}$	$n_{2;14}$	$r_{d_1b_2} + r_{d_2b_3} - r_{d_1b_20\|0d_2b_3}$
$n_{1;15}$	$r_{c_10\|a_2c_2} - r_{c_1} - r_{c_2}$	$n_{2;15}$	$r_{d_1b_2\|0d_2} - r_{d_1} - r_{d_2}$
$n_{1;16}$	$r_{a_3} + r_{a_2c_20\|0a_3c_3} - r_{a_3c_3} - r_{a_2c_2\|0a_3}$	$n_{2;16}$	$r_{b_3} + r_{b_20\|d_2b_3\|0d_3} - r_{b_3\|d_3} - r_{b_20\|d_2b_3}$
$n_{1;17}$	$r_{a_2c_2} + r_{a_3c_3} - r_{a_2c_20\|0a_3c_3}$	$n_{2;17}$	$\begin{matrix} r_{b_2\|d_2} + r_{b_3\|d_3} \\ -r_{b_20\|d_2b_3\|0d_3}\end{matrix}$
$n_{1;18}$	$r_{a_2c_2\|0a_3} - r_{a_2} - r_{a_3}$	$n_{2;18}$	$r_{b_20\|d_2b_3} - r_{b_2} - r_{b_3}$
$n_{1;19}$	$r_{c_3} + r_{c_2\|a_3} - r_{c_20\|a_3c_3}$	$n_{2;19}$	$r_{d_3} + r_{d_2b_3} - r_{d_2b_3\|0d_3}$
$n_{1;20}$	$r_{c_20\|a_3c_3} - r_{c_2} - r_{c_3}$	$n_{2;20}$	$r_{d_2b_3\|0d_3} - r_{d_2} - r_{d_3}$
$n_{1;21}$	$r_{a_3c_3} - r_{a_3}$	$n_{2;21}$	$r_{b_3\|d_3} - r_{b_3}$
l_i	$r_{0b_i0\|a_ie_ic_i\|0d_i0} - r_{a_ic_i} - r_{b_i\|d_i}$		

Proof. It follows from $\Delta_1, \ldots, \Delta_6, \Omega_1, \ldots, \Omega_6$ in (13) that

$$M\begin{pmatrix} n_{1;1} \\ n_{1;2} \\ \vdots \\ n_{1;21}\end{pmatrix} = f_1, \qquad M\begin{pmatrix} n_{2;1} \\ n_{2;2} \\ \vdots \\ n_{2;21}\end{pmatrix} = f_2, \tag{17}$$

where

$$f_1 = \begin{pmatrix} r_{a_1} \\ r_{c_1} \\ r_{a_2} \\ r_{c_2} \\ r_{a_3} \\ r_{c_3} \\ r_{a_1c_1} \\ r_{c_1|a_2} \\ r_{a_2c_2} \\ r_{c_2|a_3} \\ r_{a_3c_3} \\ r_{a_1c_1|0a_2} \\ r_{c_10|a_2c_2} \\ r_{a_2c_2|0a_3} \\ r_{c_20|a_3c_3} \\ r_{a_1c_10|0a_2c_2} \\ r_{c_10|a_2c_2|0a_3} \\ r_{a_2c_20|0a_3c_3} \\ r_{a_1c_10|0a_2c_2|00a_3} \\ r_{c_100|a_2c_20|0a_3c_3} \\ r_{a_1c_100|0a_2c_20|00a_3c_3} \end{pmatrix}, \quad f_2 = \begin{pmatrix} r_{b_1} \\ r_{d_1} \\ r_{b_2} \\ r_{d_2} \\ r_{b_3} \\ r_{d_3} \\ r_{b_1|d_1} \\ r_{d_1b_2} \\ r_{b_2|d_2} \\ r_{d_2b_3} \\ r_{b_3|d_3} \\ r_{b_10|d_1b_2} \\ r_{d_1b_2|0d_2} \\ r_{b_20|d_2b_3} \\ r_{d_2b_3|0d_3} \\ r_{b_10|d_1b_2|0d_2} \\ r_{d_1b_20|0d_2b_3} \\ r_{b_20|d_2b_3|0d_3} \\ r_{b_100|d_1b_20|0d_2b_3} \\ r_{d_1b_20|0d_2b_3|00d_3} \\ r_{b_100|d_1b_20|0d_2b_3|00d_3} \end{pmatrix},$$

$$M = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&1&1&1&1&0&1&1&1&1&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1&1&0&0&1&1&1&0&1&1&1&0&0&0&0 \\ 1&1&0&0&0&0&1&1&0&0&0&1&1&0&0&1&1&0&1&1&0& \\ 1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1& \\ 1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1&1&0&0&1&1&1&0&1&1&1&1&1&0&0 \\ 1&1&0&0&0&0&1&1&0&0&0&1&1&0&0&1&1&0&1&1&0& \\ 1&1&0&0&0&0&1&1&0&0&0&1&1&0&0&1&1&0&1&1&1& \\ 2&2&2&2&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&2&1&1&0&2&2&2&1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 2&2&1&1&0&0&2&2&1&1&0&2&2&1&1&1&1&1&1&1&0& \\ 2&1&1&0&0&0&2&1&1&0&0&2&1&1&0&2&1&1&1&1&1& \\ 2&2&2&2&1&1&2&2&2&1&1&1&1&1&1&1&1&1&0&0&0& \\ 2&2&2&1&1&0&2&2&2&1&1&2&2&1&1&1&1&1&1&1&0& \\ 2&2&1&1&0&0&2&2&1&1&0&2&2&1&1&2&1&1&1&1&1& \\ 3&3&2&2&1&1&2&2&2&1&1&2&2&1&1&1&1&1&1&1&0& \\ 3&2&2&1&1&0&3&2&2&1&1&2&2&1&1&2&1&1&1&1&1& \\ 3&3&2&2&1&1&3&2&2&1&1&2&2&1&1&2&1&1&1&1&1& \end{pmatrix}.$$

We can obtain the expressions of $n_{j;1}, \ldots, n_{j;21}$ via the linear equations (17). It is easy to verify that $l_i = r_{0b_i0|a_ie_ic_i|0d_i0} - r_{a_ic_i} - r_{b_i|d_i}$. □

Remark 2.5. The expressions of $n_{j;1}, \ldots, n_{j;21}, l_i$ play an important role in investigating the range of ranks of the general solutions to the system (1).

Let the matrices A_3, B_3, C_3, D_3, and E_3 vanish in Theorem 2.2 and Theorem 2.4, we can obtain the simultaneous decomposition for ten matrices: $A_i \in \mathbb{C}^{p_i \times t_i}$, $B_i \in \mathbb{C}^{s_i \times q_i}$, $C_i \in \mathbb{C}^{p_i \times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1} \times q_i}$, and $E_i \in \mathbb{C}^{p_i \times q_i}$ $(i = 1, 2)$.

Corollory 2.6. *Consider a set of 10 matrices: $A_i \in \mathbb{C}^{p_i \times t_i}$, $B_i \in \mathbb{C}^{s_i \times q_i}$, $C_i \in \mathbb{C}^{p_i \times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1} \times q_i}$, and $E_i \in \mathbb{C}^{p_i \times q_i}$ $(i = 1, 2)$. Then there exist*

—Unitary matrices $T_1 \in \mathbb{C}^{t_1 \times t_1}$ and $S_1 \in \mathbb{C}^{s_1 \times s_1}$.

— Nonsingular matrices $P_i \in \mathbb{C}^{p_i \times p_i}$, $T_2 \in \mathbb{C}^{t_2 \times t_2}$, $S_2 \in \mathbb{C}^{s_2 \times s_2}$, and $Q_i \in \mathbb{C}^{q_i \times q_i}$, such that

$$P_1 A_1 T_1 = \Delta_1^1,\ P_1 C_1 T_2 = \Delta_2^1,\ P_2 A_2 T_2 = \Delta_3^1,\ P_2 C_2 T_3 = \Delta_4^1,\ P_1 E_1 Q_1 = \Phi_1,$$
$$S_1 B_1 Q_1 = (\Delta_1^2)^*,\ S_2 D_1 Q_1 = (\Delta_2^2)^*,\ S_2 B_2 Q_2 = (\Delta_3^2)^*,\ S_3 D_2 Q_2 = (\Delta_4^2)^*,$$
$$P_2 E_2 Q_2 = \Phi_2,$$

where

$$\Phi_i = \begin{pmatrix} E_{11}^i & \cdots & E_{17}^i & E_{18}^i & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ E_{71}^i & \cdots & E_{77}^i & E_{78}^i & 0 \\ E_{81}^i & \cdots & E_{87}^i & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_{l_i} \end{pmatrix},$$

$$\begin{pmatrix} \Delta_1^i & \Delta_2^i \\ & \Delta_3^i \quad \Delta_4^i \end{pmatrix} = \begin{array}{c} n_{i;1} \\ n_{i;2} \\ n_{i;3} \\ n_{i;4} \\ n_{i;5} \\ n_{i;6} \\ n_{i;7} \\ l_1 \\ \\ n_{i;1} \\ n_{i;2} \\ n_{i;5} \\ n_{i;6} \\ n_{i;8} \\ n_{i;9} \\ n_{i;10} \\ \\ l_2 \end{array} \begin{pmatrix} I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ & & & & & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ & & & & & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The block rows of Φ_1 and Φ_2 are the same as the block rows of Δ_1^1 and Δ_3^1, respectively. The block columns of Φ_1 and Φ_2 are the same as the block columns of $(\Delta_1^1)^*$ and $(\Delta_3^1)^*$, respectively. The block dimensions $n_{i;1}, \ldots, n_{i;10}, l_i$ are given by:

$n_{1;1}$	$r_{a_1c_1\|0a_2} + r_{c_10\|a_2c_2} - r_{c_1\|a_2} - r_{a_1c_10\|0a_2c_2}$	$n_{2;1}$	$r_{b_10\|d_1b_2} + r_{d_1b_2\|0d_2} - r_{d_1b_2} - r_{b_10\|d_1b_2\|0d_2}$
$n_{1;2}$	$r_{c_1} + r_{a_1c_10\|0a_2c_2} - r_{a_1c_1} - r_{c_10\|a_2c_2}$	$n_{2;2}$	$r_{d_1} + r_{b_10\|d_1b_2\|0d_2} - r_{b_1\|d_1} - r_{d_1b_2\|0d_2}$
$n_{1;3}$	$r_{a_1} + r_{c_1\|a_2} - r_{a_1c_1\|0a_2}$	$n_{2;3}$	$r_{b_1} + r_{d_1b_2} - r_{b_10\|d_1b_2}$
$n_{1;4}$	$r_{a_1c_1} - r_{c_1}$	$n_{2;4}$	$r_{b_1\|d_1} - r_{d_1}$
$n_{1;5}$	$r_{a_2} + r_{a_1c_10\|0a_2c_2} - r_{a_1c_1\|0a_2} - r_{a_2c_2}$	$n_{2;5}$	$r_{b_2} + r_{b_10\|d_1b_2\|0d_2} - r_{b_10\|d_1b_2} - r_{b_2\|d_2}$
$n_{1;6}$	$r_{a_1c_1} + r_{a_2c_2} - r_{a_1c_10\|0a_2c_2}$	$n_{2;6}$	$r_{b_1\|d_1} + r_{b_2\|d_2} - r_{b_10\|d_1b_2\|0d_2}$
$n_{1;7}$	$r_{a_1c_1\|0a_2} - r_{a_1} - r_{a_2}$	$n_{2;7}$	$r_{b_10\|d_1b_2} - r_{b_1} - r_{b_2}$
$n_{1;8}$	$r_{c_2} + r_{c_1\|a_2} - r_{c_10\|a_2c_2}$	$n_{2;8}$	$r_{d_2} + r_{d_1b_2} - r_{d_1b_2\|0d_2}$
$n_{1;9}$	$r_{c_10\|a_2c_2} - r_{c_1} - r_{c_2}$	$n_{2;9}$	$r_{d_1b_2\|0d_2} - r_{d_1} - r_{d_2}$
$n_{1;10}$	$r_{a_2c_2} - r_{a_2}$	$n_{2;10}$	$r_{b_2\|d_2} - r_{b_2}$
l_i	$r_{0b_i0\|a_ie_ic_i\|0d_i0} - r_{a_ic_i} - r_{b_i\|d_i}$		

3. The system of two-sided coupled generalized Sylvester matrix equations (1)

In this section, we give some applications of the simultaneous decomposition for the matrix array (8). Using this simultaneous decomposition, we can:

- §3.1. Give some necessary and sufficient conditions for the existence of a solution $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1} \times \mathbb{C}^{t_2\times s_2} \times \mathbb{C}^{t_3\times s_3} \times \mathbb{C}^{t_4\times s_4}$ to the system (1).
- §3.2. Give the general solutions to the system (1) when the solvability conditions are satisfied.
- §3.3. Give the range of ranks of X_1 and X_4 in the system (1), respectively.

3.1. Solvability conditions to the system of two-sided coupled generalized Sylvester matrix equations (1)

In this section, we give some necessary and sufficient conditions for the existence of a solution to the system (1) through the canonical form (16).

Theorem 3.1. *The system of two-sided coupled generalized Sylvester matrix equations (1) is consistent if and only if the ranks satisfy:*

$$r_{a_ie_ic_i} = r_{a_ic_i},\ r_{b_i|e_i|d_i} = r_{b_i|d_i},\ r_{a_ie_i|0d_i} = r_{a_i} + r_{d_i},\ r_{b_i0|e_ic_i} = r_{b_i} + r_{c_i} \quad (i = 1, 2, 3), \tag{18}$$

$$r_{a_je_jc_j00|0d_j0b_{j+1}0|00a_{j+1}-e_{j+1}c_{j+1}} = r_{a_jc_j0|0a_{j+1}c_{j+1}} + r_{d_jb_{j+1}}, \tag{19}$$

$$r_{b_j00|e_jc_j0|d_j0b_{j+1}|0a_{j+1}-e_{j+1}|00d_{j+1}} = r_{c_j|a_{j+1}} + r_{b_j0|d_jb_{j+1}|0d_{j+1}}, \tag{20}$$

$$r_{a_je_jc_j0|0d_j0b_{j+1}|00a_{j+1}-e_{j+1}|000d_{j+1}} = r_{a_jc_j|0a_{j+1}} + r_{d_jb_{j+1}|0d_{j+1}}, \tag{21}$$

$$r_{b_j000|e_jc_j00|d_j0b_{j+1}0|0a_{j+1}-e_{j+1}c_{j+1}} = r_{c_j0|a_{j+1}c_{j+1}} + r_{b_j0|d_jb_{j+1}},\ j = 1, 2, \tag{22}$$

$$r_{a_1e_1c_1 0000|0d_1 0b_2 000|00a_2 - e_2c_2 00|000d_2 0b_3 0|0000a_3e_3c_3} = r_{a_1c_1 00|0a_2c_2 0|00a_3c_3} + r_{d_1b_2 0|0d_2b_3}, \tag{23}$$

$$r_{b_1 0000|e_1c_1 000|d_1 0b_2 00|0a_2 - e_2c_2 0|00d_2 0b_3|000a_3e_3|0000d_3} = r_{c_1 0|a_2c_2|0a_3} + r_{b_1 00|d_1b_2 0|0d_2b_3|00d_3}, \tag{24}$$

$$r_{a_1e_1c_1 000|0d_1 0b_2 00|00a_2 - e_2c_2 0|000d_2 0b_3|0000a_3e_3|00000d_3} = r_{a_1c_1 0|0a_2c_2|00a_3} + r_{d_1b_2 0|0d_2b_3|00d_3}, \tag{25}$$

$$r_{b_1 00000|e_1c_1 00000|d_1 0b_2 000|0a_2 - e_2c_2 00|00d_2 0b_3 0|000a_3e_3c_3} = r_{c_1 00|a_2c_2 0|0a_3c_3} + r_{b_1 00|d_1b_2 0|0d_2b_3}. \tag{26}$$

Proof. *(only if)-part* Suppose that the system (1) has a solution $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1} \times \mathbb{C}^{t_2\times s_2} \times \mathbb{C}^{t_3\times s_3} \times \mathbb{C}^{t_4\times s_4}$, say, $(X_1^0, X_2^0, X_3^0, X_4^0)$. Then, we have

$$A_iX_i^0B_i + C_iX_{i+1}^0D_i = E_i \quad (i = 1, 2, 3). \tag{27}$$

Now we want to show that the rank equalities in (18)–(26) hold using the matrix equations (27) and elementary matrix operations.

- For the equalities in (18): Note that $A_iX_i^0B_i + C_iX_{i+1}^0D_i = E_i$ $(i = 1, 2, 3)$. Then, we have

$$r_{a_ie_ic_i} = r(A_i, E_i, C_i) = r(A_i, A_iX_i^0B_i + C_iX_{i+1}^0D_i, C_i) = r(A_i, C_i) = r_{a_ic_i}.$$

  Similarly, we can prove the other three equalities in (18).
- For the equalities in (19)–(26): Observe that

$$\begin{aligned} r_{a_je_jc_j 00|0d_j 0b_{j+1}0|00a_{j+1} - e_{j+1}c_{j+1}} &= r\begin{pmatrix} A_j & E_j & C_j & 0 & 0 \\ 0 & D_j & 0 & B_{j+1} & 0 \\ 0 & 0 & A_{j+1} & -E_{j+1} & C_{j+1} \end{pmatrix} \\ &= r\begin{pmatrix} A_j & A_jX_j^0B_j + C_jX_{j+1}^0D_j & C_j & 0 & 0 \\ 0 & D_j & 0 & B_{j+1} & 0 \\ 0 & 0 & A_{j+1} & -A_{j+1}X_{j+1}^0B_{j+1} - C_{j+1}X_{j+2}^0D_{j+1} & C_{j+1} \end{pmatrix} \\ &= r\begin{pmatrix} A_j & 0 & C_j & 0 & 0 \\ 0 & D_j & 0 & B_{j+1} & 0 \\ 0 & 0 & A_{j+1} & 0 & C_{j+1} \end{pmatrix} = r_{a_jc_j 0|0a_{j+1}c_{j+1}} + r_{d_jb_{j+1}}. \end{aligned}$$

  Similarly, we can prove the equalities in (20)–(26).

(if)-part. By Theorem 2.2, we know that the system (1) has a solution $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1} \times \mathbb{C}^{t_2\times s_2} \times \mathbb{C}^{t_3\times s_3} \times \mathbb{C}^{t_4\times s_4}$ is equivalent to the system (16), where $T_k, S_k, \Phi_i, \Delta_{2i-1}, \Delta_{2i}, \Omega_{2i-1}, \Omega_{2i}$ are defined in Theorem 2.2 $(k = 1, 2, 3, 4)$. Let the matrices

$$\begin{cases} Y_k := \begin{pmatrix} Y_{11}^k & \cdots & Y_{17}^k \\ \vdots & \ddots & \vdots \\ Y_{71}^k & \cdots & Y_{77}^k \end{pmatrix}, & \text{if } k = 1, 4 \\ Y_k := \begin{pmatrix} Y_{11}^k & \cdots & Y_{1,15}^k \\ \vdots & \ddots & \vdots \\ Y_{15,1}^k & \cdots & Y_{15,15}^k \end{pmatrix}, & \text{if } k = 2, 3 \end{cases}$$

be partitioned in accordance with (16). Substituting Y_k into the matrix equations in (16) yields

$$\begin{pmatrix} Y_{11}^1+Y_{11}^2 & Y_{12}^1+Y_{12}^2 & Y_{13}^1+Y_{13}^2 & Y_{14}^1+Y_{14}^2 & Y_{15}^1+Y_{15}^2 & Y_{16}^1 & Y_{16}^2 & Y_{17}^2 & Y_{18}^2 & Y_{19}^2 & Y_{1,10}^2 & 0\ 0 \\ Y_{21}^1+Y_{12}^2 & Y_{22}^1+Y_{22}^2 & Y_{23}^1+Y_{23}^2 & Y_{24}^1+Y_{24}^2 & Y_{25}^1+Y_{25}^2 & Y_{26}^1 & Y_{26}^2 & Y_{27}^2 & Y_{28}^2 & Y_{29}^2 & Y_{2,10}^2 & 0\ 0 \\ Y_{31}^1+Y_{13}^2 & Y_{32}^1+Y_{23}^2 & Y_{33}^1+Y_{33}^2 & Y_{34}^1+Y_{34}^2 & Y_{35}^1+Y_{35}^2 & Y_{36}^1 & Y_{36}^2 & Y_{37}^2 & Y_{38}^2 & Y_{39}^2 & Y_{3,10}^2 & 0\ 0 \\ Y_{41}^1+Y_{14}^2 & Y_{42}^1+Y_{24}^2 & Y_{34}^1+Y_{43}^2 & Y_{44}^1+Y_{44}^2 & Y_{45}^1+Y_{45}^2 & Y_{46}^1 & Y_{46}^2 & Y_{47}^2 & Y_{48}^2 & Y_{49}^2 & Y_{4,10}^2 & 0\ 0 \\ Y_{51}^1+Y_{15}^2 & Y_{52}^1+Y_{25}^2 & Y_{35}^1+Y_{53}^2 & Y_{54}^1+Y_{54}^2 & Y_{55}^1+Y_{55}^2 & Y_{56}^1 & Y_{56}^2 & Y_{57}^2 & Y_{58}^2 & Y_{59}^2 & Y_{5,10}^2 & 0\ 0 \\ Y_{61}^1 & Y_{62}^1 & Y_{63}^1 & Y_{64}^1 & Y_{65}^1 & Y_{66}^1 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \\ Y_{61}^2 & Y_{62}^2 & Y_{63}^2 & Y_{64}^2 & Y_{65}^2 & 0 & Y_{66}^2 & Y_{67}^2 & Y_{68}^2 & Y_{69}^2 & Y_{6,10}^2 & 0\ 0 \\ Y_{71}^2 & Y_{72}^2 & Y_{73}^2 & Y_{74}^2 & Y_{75}^2 & 0 & Y_{76}^2 & Y_{77}^2 & Y_{78}^2 & Y_{79}^2 & Y_{7,10}^2 & 0\ 0 \\ Y_{81}^2 & Y_{82}^2 & Y_{83}^2 & Y_{84}^2 & Y_{85}^2 & 0 & Y_{86}^2 & Y_{87}^2 & Y_{88}^2 & Y_{89}^2 & Y_{8,10}^2 & 0\ 0 \\ Y_{91}^2 & Y_{92}^2 & Y_{93}^2 & Y_{94}^2 & Y_{95}^2 & 0 & Y_{96}^2 & Y_{97}^2 & Y_{98}^2 & Y_{99}^2 & Y_{9,10}^2 & 0\ 0 \\ Y_{10,1}^2 & Y_{10,2}^2 & Y_{10,3}^2 & Y_{10,4}^2 & Y_{10,5}^2 & 0 & Y_{10,6}^2 & Y_{10,7}^2 & Y_{10,8}^2 & Y_{10,9}^2 & Y_{10,10}^2 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \end{pmatrix} = \Phi_1, \tag{28}$$

$$(\Phi_2^1,\ \Phi_2^2) = \Phi_2, \tag{29}$$

$$\begin{pmatrix} Y_{11}^3+Y_{11}^4 & Y_{12}^3 & Y_{14}^3+Y_{12}^4 & Y_{15}^3 & Y_{17}^3+Y_{13}^4 & Y_{18}^3 & Y_{1,10}^3+Y_{14}^4 & Y_{1,11}^3 & Y_{1,13}^3+Y_{15}^4 & Y_{1,14}^3 & Y_{16}^4 & 0\ 0 \\ Y_{21}^3 & Y_{22}^3 & Y_{24}^3 & Y_{25}^3 & Y_{27}^3 & Y_{28}^3 & Y_{2,10}^3 & Y_{2,11}^3 & Y_{2,13}^3 & Y_{2,14}^3 & 0 & 0\ 0 \\ Y_{41}^3+Y_{21}^4 & Y_{42}^3 & Y_{44}^3+Y_{22}^4 & Y_{45}^3 & Y_{47}^3+Y_{23}^4 & Y_{48}^3 & Y_{4,10}^3+Y_{24}^4 & Y_{4,11}^3 & Y_{4,13}^3+Y_{25}^4 & Y_{4,14}^3 & Y_{26}^4 & 0\ 0 \\ Y_{51}^3 & Y_{52}^3 & Y_{54}^3 & Y_{55}^3 & Y_{57}^3 & Y_{58}^3 & Y_{5,10}^3 & Y_{5,11}^3 & Y_{5,13}^3 & Y_{5,14}^3 & 0 & 0\ 0 \\ Y_{71}^3+Y_{31}^4 & Y_{72}^3 & Y_{74}^3+Y_{32}^4 & Y_{75}^3 & Y_{77}^3+Y_{33}^4 & Y_{78}^3 & Y_{7,10}^3+Y_{34}^4 & Y_{7,11}^3 & Y_{7,13}^3+Y_{35}^4 & Y_{7,14}^3 & Y_{36}^4 & 0\ 0 \\ Y_{81}^3 & Y_{82}^3 & Y_{84}^3 & Y_{85}^3 & Y_{87}^3 & Y_{88}^3 & Y_{8,10}^3 & Y_{8,11}^3 & Y_{8,13}^3 & Y_{8,14}^3 & 0 & 0\ 0 \\ Y_{10,1}^3+Y_{41}^4 & Y_{10,2}^3 & Y_{10,4}^3+Y_{42}^4 & Y_{10,5}^3 & Y_{10,7}^3+Y_{43}^4 & Y_{10,8}^3 & Y_{10,10}^3+Y_{44}^4 & Y_{10,11}^3 & Y_{10,13}^3+Y_{45}^4 & Y_{10,14}^3 & Y_{46}^4 & 0\ 0 \\ Y_{11,1}^3 & Y_{11,2}^3 & Y_{11,4}^3 & Y_{11,5}^3 & Y_{11,7}^3 & Y_{11,8}^3 & Y_{11,10}^3 & Y_{11,11}^3 & Y_{11,13}^3 & Y_{11,14}^3 & 0 & 0\ 0 \\ Y_{13,1}^3+Y_{51}^4 & Y_{13,2}^3 & Y_{13,4}^3+Y_{52}^4 & Y_{13,5}^3 & Y_{13,7}^3+Y_{53}^4 & Y_{13,8}^3 & Y_{13,10}^3+Y_{54}^4 & Y_{13,11}^3 & Y_{13,13}^3+Y_{55}^4 & Y_{13,14}^3 & Y_{56}^4 & 0\ 0 \\ Y_{14,1}^3 & Y_{14,2}^3 & Y_{14,4}^3 & Y_{14,5}^3 & Y_{14,7}^3 & Y_{14,8}^3 & Y_{14,10}^3 & Y_{14,11}^3 & Y_{14,13}^3 & Y_{14,14}^3 & 0 & 0\ 0 \\ Y_{61}^4 & 0 & Y_{62}^4 & 0 & Y_{63}^4 & 0 & Y_{64}^4 & 0 & Y_{65}^4 & 0 & Y_{66}^4 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \end{pmatrix} = \Phi_3, \tag{30}$$

where

$$\Phi_2^1 = \begin{pmatrix}
Y_{11}^2+Y_{11}^3 & Y_{12}^2+Y_{12}^3 & Y_{13}^2+Y_{13}^3 & Y_{14}^2 & Y_{16}^2+Y_{14}^3 & Y_{17}^2+Y_{15}^3 & Y_{18}^2+Y_{16}^3 \\
Y_{21}^2+Y_{21}^3 & Y_{22}^2+Y_{22}^3 & Y_{23}^2+Y_{23}^3 & Y_{24}^2 & Y_{26}^2+Y_{24}^3 & Y_{27}^2+Y_{25}^3 & Y_{28}^2+Y_{26}^3 \\
Y_{31}^2+Y_{31}^3 & Y_{32}^2+Y_{32}^3 & Y_{33}^2+Y_{33}^3 & Y_{34}^2 & Y_{36}^2+Y_{34}^3 & Y_{37}^2+Y_{35}^3 & Y_{38}^2+Y_{36}^3 \\
Y_{41}^2 & Y_{42}^2 & Y_{43}^2 & Y_{44}^2 & Y_{46}^2 & Y_{47}^2 & Y_{48}^2 \\
Y_{61}^2+Y_{41}^3 & Y_{62}^2+Y_{42}^3 & Y_{63}^2+Y_{43}^3 & Y_{64}^2 & Y_{66}^2+Y_{44}^3 & Y_{67}^2+Y_{45}^3 & Y_{68}^2+Y_{46}^3 \\
Y_{71}^2+Y_{51}^3 & Y_{72}^2+Y_{52}^3 & Y_{73}^2+Y_{53}^3 & Y_{74}^2 & Y_{76}^2+Y_{54}^3 & Y_{77}^2+Y_{55}^3 & Y_{78}^2+Y_{56}^3 \\
Y_{81}^2+Y_{61}^3 & Y_{82}^2+Y_{62}^3 & Y_{83}^2+Y_{63}^3 & Y_{84}^2 & Y_{86}^2+Y_{64}^3 & Y_{87}^2+Y_{65}^3 & Y_{88}^2+Y_{66}^3 \\
Y_{91}^2 & Y_{92}^2 & Y_{93}^2 & Y_{94}^2 & Y_{96}^2 & Y_{97}^2 & Y_{98}^2 \\
Y_{11,1}^2+Y_{71}^3 & Y_{11,2}^2+Y_{72}^3 & Y_{11,3}^2+Y_{73}^3 & Y_{11,4}^2 & Y_{11,6}^2+Y_{74}^3 & Y_{11,7}^2+Y_{75}^3 & Y_{11,8}^2+Y_{76}^3 \\
Y_{12,1}^2+Y_{81}^3 & Y_{12,2}^2+Y_{82}^3 & Y_{12,3}^2+Y_{83}^3 & Y_{12,4}^2 & Y_{12,6}^2+Y_{84}^3 & Y_{12,7}^2+Y_{85}^3 & Y_{12,8}^2+Y_{86}^3 \\
Y_{13,1}^2+Y_{91}^3 & Y_{13,2}^2+Y_{92}^3 & Y_{13,3}^2+Y_{93}^3 & Y_{13,4}^2 & Y_{13,6}^2+Y_{94}^3 & Y_{13,7}^2+Y_{95}^3 & Y_{13,8}^2+Y_{96}^3 \\
Y_{14,1}^2 & Y_{14,2}^2 & Y_{14,3}^2 & Y_{14,4}^2 & Y_{14,6}^2 & Y_{14,7}^2 & Y_{14,8}^2 \\
Y_{10,1}^3 & Y_{10,2}^3 & Y_{10,3}^3 & 0 & Y_{10,4}^3 & Y_{10,5}^3 & Y_{10,6}^3 \\
Y_{11,1}^3 & Y_{11,2}^3 & Y_{11,3}^3 & 0 & Y_{11,4}^3 & Y_{11,5}^3 & Y_{11,6}^3 \\
Y_{12,1}^3 & Y_{12,2}^3 & Y_{12,3}^3 & 0 & Y_{12,4}^3 & Y_{12,5}^3 & Y_{12,6}^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_2^2 = \begin{pmatrix}
Y_{19}^2 & Y_{1,11}^2+Y_{17}^3 & Y_{1,12}^2+Y_{18}^3 & Y_{1,13}^2+Y_{19}^3 & Y_{1,14}^2 & Y_{1,10}^3 & Y_{1,11}^3 & Y_{1,12}^3 & 0 & 0 \\
Y_{29}^2 & Y_{2,11}^2+Y_{27}^3 & Y_{2,12}^2+Y_{28}^3 & Y_{2,13}^2+Y_{29}^3 & Y_{2,14}^2 & Y_{2,10}^3 & Y_{2,11}^3 & Y_{2,12}^3 & 0 & 0 \\
Y_{39}^2 & Y_{3,11}^2+Y_{37}^3 & Y_{3,12}^2+Y_{38}^3 & Y_{3,13}^2+Y_{39}^3 & Y_{3,14}^2 & Y_{3,10}^3 & Y_{3,11}^3 & Y_{3,12}^3 & 0 & 0 \\
Y_{49}^2 & Y_{4,11}^2 & Y_{4,12}^2 & Y_{4,13}^2 & Y_{4,14}^2 & 0 & 0 & 0 & 0 & 0 \\
Y_{69}^2 & Y_{6,11}^2+Y_{47}^3 & Y_{6,12}^2+Y_{48}^3 & Y_{6,13}^2+Y_{49}^3 & Y_{6,14}^2 & Y_{4,10}^3 & Y_{4,11}^3 & Y_{4,12}^3 & 0 & 0 \\
Y_{79}^2 & Y_{7,11}^2+Y_{57}^3 & Y_{7,12}^2+Y_{58}^3 & Y_{7,13}^2+Y_{59}^3 & Y_{7,14}^2 & Y_{5,10}^3 & Y_{5,11}^3 & Y_{5,12}^3 & 0 & 0 \\
Y_{89}^2 & Y_{8,11}^2+Y_{67}^3 & Y_{8,12}^2+Y_{68}^3 & Y_{8,13}^2+Y_{69}^3 & Y_{8,14}^2 & Y_{6,10}^3 & Y_{6,11}^3 & Y_{6,12}^3 & 0 & 0 \\
Y_{99}^2 & Y_{9,11}^2 & Y_{9,12}^2 & Y_{9,13}^2 & Y_{9,14}^2 & 0 & 0 & 0 & 0 & 0 \\
Y_{11,9}^2 & Y_{11,11}^2+Y_{77}^3 & Y_{11,12}^2+Y_{78}^3 & Y_{11,13}^2+Y_{79}^3 & Y_{11,14}^2 & Y_{7,10}^3 & Y_{7,11}^3 & Y_{7,12}^3 & 0 & 0 \\
Y_{12,9}^2 & Y_{12,11}^2+Y_{87}^3 & Y_{12,12}^2+Y_{88}^3 & Y_{12,13}^2+Y_{89}^3 & Y_{12,14}^2 & Y_{8,10}^3 & Y_{8,11}^3 & Y_{8,12}^3 & 0 & 0 \\
Y_{13,9}^2 & Y_{13,11}^2+Y_{97}^3 & Y_{13,12}^2+Y_{98}^3 & Y_{13,13}^2+Y_{99}^3 & Y_{13,14}^2 & Y_{9,10}^3 & Y_{9,11}^3 & Y_{9,12}^3 & 0 & 0 \\
Y_{14,9}^2 & Y_{14,11}^2 & Y_{14,12}^2 & Y_{14,13}^2 & Y_{14,14}^2 & 0 & 0 & 0 & 0 & 0 \\
0 & Y_{10,7}^3 & Y_{10,8}^3 & Y_{10,9}^3 & 0 & Y_{10,10}^3 & Y_{10,11}^3 & Y_{10,12}^3 & 0 & 0 \\
0 & Y_{11,7}^3 & Y_{11,8}^3 & Y_{11,9}^3 & 0 & Y_{11,10}^3 & Y_{11,11}^3 & Y_{11,12}^3 & 0 & 0 \\
0 & Y_{12,7}^3 & Y_{12,8}^3 & Y_{12,9}^3 & 0 & Y_{12,10}^3 & Y_{12,11}^3 & Y_{12,12}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Upon computation, we obtain that the matrix equations (28)–(30) are consistent if and only if

$$l_i = 0, \quad (i = 1, 2, 3), \quad E_{12,v}^1 = 0, \quad E_{12,v}^3 = 0, \quad E_{16,u}^2 = 0, \tag{31}$$

$$E_{v,12}^1 = 0, \quad E_{v,12}^3 = 0, \quad E_{u,16}^2 = 0, \quad (v = 1, \ldots, 11), \quad (u = 1, \ldots, 15), \tag{32}$$

$$\begin{cases} E_{76}^1 = 0, E_{86}^1 = 0, E_{96}^1 = 0, E_{10,6}^1 = 0, E_{11,6}^1 = 0, \\ E_{11,2}^3 = 0, E_{11,4}^3 = 0, E_{11,6}^3 = 0, E_{11,8}^3 = 0, E_{11,10}^3 = 0, \end{cases} \tag{33}$$

$$\begin{cases} E_{13,4}^2 = 0, E_{13,8}^2 = 0, E_{13,12}^2 = 0, E_{14,4}^2 = 0, \\ E_{14,8}^2 = 0, E_{14,12}^2 = 0, E_{15,4}^2 = 0, E_{15,8}^2 = 0, E_{15,12}^2 = 0, \end{cases} \tag{34}$$

$$\begin{cases} E^1_{67}=0, E^1_{68}=0, E^1_{69}=0, E^1_{6,10}=0, E^1_{6,11}=0, \\ E^3_{2,11}=0, E^3_{4,11}=0, E^3_{6,11}=0, E^3_{8,11}=0, E^3_{10,11}=0, \end{cases} \tag{35}$$

$$\begin{cases} E^2_{4,13}=0, E^2_{4,14}=0, E^2_{4,15}=0, E^2_{8,13}=0, \\ E^2_{8,14}=0, E^2_{8,15}=0, E^2_{12,13}=0, E^2_{12,14}=0, E^2_{12,15}=0, \end{cases} \tag{36}$$

$$\begin{cases} E^1_{10,1}=E^2_{81}, E^1_{10,2}=E^2_{82}, E^1_{10,3}=E^2_{83}, E^1_{10,4}=E^2_{84}, \\ E^1_{10,7}=E^2_{85}, E^1_{10,8}=E^2_{86}, E^1_{10,9}=E^2_{87}, E^1_{10,10}=E^2_{88}, \end{cases} \tag{37}$$

$$E^1_{1,10}=E^2_{18}, E^1_{2,10}=E^2_{28}, E^1_{3,10}=E^2_{38}, E^1_{4,10}=E^2_{48}, E^1_{7,10}=E^2_{58}, E^1_{8,10}=E^2_{68},$$
$$E^1_{9,10}=E^2_{78}, \tag{38}$$
$$E^1_{74}=E^2_{54}, E^1_{84}=E^2_{64}, E^1_{94}=E^2_{74}, E^1_{47}=E^2_{45}, E^1_{48}=E^2_{46}, E^1_{49}=E^2_{47}, \tag{39}$$

$$\begin{cases} E^2_{14,1}=E^3_{81}, E^2_{14,2}=E^3_{82}, E^2_{14,5}=E^3_{83}, E^2_{14,6}=E^3_{84}, \\ E^2_{14,9}=E^3_{85}, E^2_{14,10}=E^3_{86}, E^2_{14,13}=E^3_{87}, E^2_{14,14}=E^3_{88}, \end{cases} \tag{40}$$

$$\begin{cases} E^2_{1,14}=E^3_{18}, E^2_{2,14}=E^3_{28}, E^2_{5,14}=E^3_{38}, E^2_{6,14}=E^3_{48}, \\ E^2_{9,14}=E^3_{58}, E^2_{10,14}=E^3_{68}, E^2_{13,14}=E^3_{78}, \end{cases} \tag{41}$$

$$E^2_{2,13}=E^3_{27}, E^2_{6,13}=E^3_{47}, E^2_{10,13}=E^3_{67}, E^2_{13,2}=E^3_{72}, E^2_{13,6}=E^3_{74}, E^2_{13,10}=E^3_{76}, \tag{42}$$
$$E^3_{41}+E^1_{81}=E^2_{61}, E^3_{42}+E^1_{82}=E^2_{62}, E^3_{43}+E^1_{87}=E^2_{65}, E^3_{44}+E^1_{88}=E^2_{66}, \tag{43}$$
$$E^3_{14}+E^1_{18}=E^2_{16}, E^3_{24}+E^1_{28}=E^2_{26}, E^3_{34}+E^1_{78}=E^2_{56}, \tag{44}$$
$$E^3_{32}+E^1_{72}=E^2_{52}, E^3_{23}+E^1_{27}=E^2_{25}. \tag{45}$$

Now we want to prove that (18)–(26) $\Longrightarrow$ (31)–(45).

- First, we prove that (18) $\Longrightarrow$ (31)–(36). From Theorem 2.2, we have

$$\begin{aligned} & r_{a_ie_ic_i}=r_{a_ic_i} \Longrightarrow r(\Phi_i,\Delta_{2i-1},\Delta_{2i})=r(\Delta_{2i-1},\Delta_{2i}) \\ & \Longrightarrow l_i=0, \quad E^1_{12,v}=0, \quad E^3_{12,v}=0, \quad E^2_{16,u}=0, \quad (v=1,\ldots,11), \quad (u=1,\ldots,15) \\ & \Longrightarrow (31). \end{aligned}$$

  Similarly, we have

$$\begin{aligned} & r_{b_i|e_i|d_i}=r_{b_i|d_i} \Longrightarrow (32), \ r_{a_ie_i|0d_i}=r_{a_i}+r_{d_i} \Longrightarrow (33) \text{ and } (34), \\ & r_{b_i0|e_ic_i}=r_{b_i}+r_{c_i} \Longrightarrow (35) \text{ and } (36). \end{aligned}$$

- Then, we show that (19)–(22) $\Longrightarrow$ (37)–(42). Case 1. If $j = 1$, then we have

$$r_{a_1e_1c_100|0d_10b_20|00a_2-e_2c_2} = r_{a_1c_10|0a_2c_2} + r_{d_1b_2}$$
$$\Longrightarrow r\begin{pmatrix} \Delta_1 & \Phi_1 & \Delta_2 & 0 & 0 \\ 0 & \Omega_2 & 0 & \Omega_3 & 0 \\ 0 & 0 & \Delta_3 & -\Phi_2 & \Delta_4 \end{pmatrix} = r\begin{pmatrix} \Delta_1 & \Delta_2 & 0 \\ 0 & \Delta_3 & \Delta_4 \end{pmatrix} + r(\Omega_2, \Omega_3) \Longrightarrow (37).$$

Similarly, we obtain

$$r_{b_100|e_1c_10|d_10b_2|0a_2-e_2|00d_2} = r_{c_1|a_2} + r_{b_10|d_1b_2|0d_2} \Longrightarrow (38),$$
$$\begin{cases} r_{a_1e_1c_10|0d_10b_2|00a_2-e_2|000d_2} = r_{a_1c_1|0a_2} + r_{d_1b_2|0d_2}, \\ r_{b_1000|e_1c_100|d_10b_20|0a_2-e_2c_2} = r_{c_10|a_2c_2} + r_{b_10|d_1b_2}, \end{cases} \Longrightarrow (39).$$

Case 2. If $j = 2$, then we have

$$r_{a_2e_2c_200|0d_20b_30|00a_3-e_3c_3} = r_{a_2c_20|0a_3c_3} + r_{d_2b_3} \Longrightarrow (40)$$
$$r_{b_200|e_2c_20|d_20b_3|0a_3-e_3|00d_3} = r_{c_2|a_3} + r_{b_20|d_2b_3|0d_3} \Longrightarrow (41),$$
$$\begin{cases} r_{a_2e_2c_20|0d_20b_3|00a_3-e_3|000d_3} = r_{a_2c_2|0a_3} + r_{d_2b_3|0d_3} \\ r_{b_2000|e_2c_200|d_20b_30|0a_3-e_3c_3} = r_{c_20|a_3c_3} + r_{b_20|d_2b_3} \end{cases} \Longrightarrow (42).$$

- Finally, we prove that (23)–(26) $\Longrightarrow$ (43)–(45). Applying the approach of the first two steps, we obtain that

$$r_{a_1e_1c_10000|0d_10b_2000|00a_2-e_2c_200|000d_20b_30|0000a_3e_3c_3} = r_{a_1c_100|0a_2c_20|00a_3c_3} + r_{d_1b_20|0d_2b_3}$$
$$\Longrightarrow (43),$$

$$r_{b_10000|e_1c_1000|d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3|0000d_3} = r_{c_10|a_2c_2|0a_3} + r_{b_100|d_1b_20|0d_2b_3|00d_3}$$
$$\Longrightarrow (44),$$

$$\begin{cases} r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3|00000d_3} = r_{a_1c_10|0a_2c_2|00a_3} + r_{d_1b_20|0d_2b_3|00d_3} \\ r_{b_100000|e_1c_100000|d_10b_2000|0a_2-e_2c_200|00d_20b_30|000a_3e_3c_3} = r_{c_100|a_2c_20|0a_3c_3} + r_{b_100|d_1b_20|0d_2b_3} \end{cases}$$
$$\Rightarrow (45). \quad \square$$

Example 3.2. Let $\mathbf{i} = \sqrt{-1}$ be the imaginary unit, and

$$A_1 = (1, \mathbf{i}), B_1 = (\mathbf{i}, 1 - \mathbf{i}), C_1 = (3 + 2\mathbf{i}, 5, 7), D_1 = \begin{pmatrix} 2+3\mathbf{i} & 2 \\ -1 & 0 \end{pmatrix}, E_1 = (54 + 90\mathbf{i}, 70 - 8\mathbf{i}),$$
$$A_2 = (1, -\mathbf{i}, \mathbf{i}), B_2 = \begin{pmatrix} 3+\mathbf{i} & 0 \\ -2 & -1 \end{pmatrix}, C_2 = (1, -\mathbf{i}), D_2 = (2 + \mathbf{i}, 3 - \mathbf{i}), E_2 = (1 - 6\mathbf{i}, -1 - 5\mathbf{i}),$$
$$A_3 = \begin{pmatrix} 1+\mathbf{i} & -\mathbf{i} \\ 2 & 3 \\ 0 & -2 \end{pmatrix}, B_3 = (1, -\mathbf{i}), C_3 = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}, D_3 = \begin{pmatrix} -3 & 2\mathbf{i} \\ 0 & -1 \end{pmatrix}, E_3 = \begin{pmatrix} -2\mathbf{i} & 1 \\ 1 & -2 \\ -2 & \mathbf{i} \end{pmatrix}.$$

Note that

$$r_{a_3e_3c_3} = 3 \neq r_{a_3c_3} = 2,\ r_{a_2e_2c_200|0d_20b_30|00a_3-e_3c_3} = 5 \neq r_{a_2c_20|0a_3c_3} + r_{d_2b_3} = 4,$$

$$r_{b_30|e_3c_3} = 3 \neq r_{b_3} + r_{c_3} = 2,\ r_{b_2000|e_2c_200|d_20b_30|0a_3-e_3c_3} = 7 \neq r_{c_20|a_3c_3} + r_{b_20|d_2b_3} = 6,$$

$$\begin{aligned} r_{a_1e_1c_10000|0d_10b_2000|00a_2-e_2c_200|000d_20b_30|0000a_3e_3c_3} &= 8 \\ \neq r_{a_1c_100|0a_2c_20|00a_3c_3} + r_{d_1b_20|0d_2b_3} &= 7, \end{aligned}$$

$$\begin{aligned} r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3|00000d_3} &= 8 \\ \neq r_{a_1c_10|0a_2c_2|00a_3} + r_{d_1b_20|0d_2b_3|00d_3} &= 7, \end{aligned}$$

$$\begin{aligned} r_{b_100000|e_1c_10000|d_10b_2000|0a_2-e_2c_200|00d_20b_30|000a_3e_3c_3} &= 9 \\ \neq r_{c_100|a_2c_20|0a_3c_3} + r_{b_100|d_1b_20|0d_2b_3} &= 8. \end{aligned}$$

Hence, the system of two-sided coupled generalized Sylvester matrix equations (1) is not consistent.

Example 3.3. Let $\mathbf{i} = \sqrt{-1}$ be the imaginary unit, and

$$A_1 = \begin{pmatrix} 1+\mathbf{i} & 2 & 3 \\ -1 & -2\mathbf{i} & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \mathbf{i} & -1 \\ 2\mathbf{i} & -2 \\ 1-\mathbf{i} & 1+\mathbf{i} \end{pmatrix}, C_1 = \begin{pmatrix} -1 & \mathbf{i} & 3-\mathbf{i} \\ -2 & 2\mathbf{i} & 6-2\mathbf{i} \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 2 & 1-\mathbf{i} \\ 5 & 2\mathbf{i} \\ -3 & 7 \end{pmatrix}, E_1 = \begin{pmatrix} 25+95\mathbf{i} & -60-28\mathbf{i} \\ 106+93\mathbf{i} & -23 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 4-\mathbf{i} & 2 \\ -3 & 2\mathbf{i} & -1 \end{pmatrix}, B_2 = \begin{pmatrix} -1 & -\mathbf{i} \\ 3 & 3\mathbf{i} \\ -1+\mathbf{i} & -1-i \end{pmatrix}, C_2 = \begin{pmatrix} -2\mathbf{i} & 3 & 1-\mathbf{i} \\ -3 & 1+\mathbf{i} & 2+3\mathbf{i} \end{pmatrix},$$
$$D_2 = \begin{pmatrix} \mathbf{i} & 2 \\ -\mathbf{i} & -2 \\ -1 & 2\mathbf{i} \end{pmatrix}, E_2 = \begin{pmatrix} -11-17\mathbf{i} & 2+19\mathbf{i} \\ 48-14\mathbf{i} & -76-27\mathbf{i} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1-\mathbf{i} & 3 & -1 \\ 1+\mathbf{i} & 4 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 3 & \mathbf{i} \\ -2 & 3 \\ 1 & -\mathbf{i} \end{pmatrix}, C_3 = \begin{pmatrix} 1+2\mathbf{i} & 2 & -1 \\ 3\mathbf{i} & 3 & -\mathbf{i} \end{pmatrix},$$
$$D_3 = \begin{pmatrix} 2\mathbf{i} & 5 \\ 1-3\mathbf{i} & \mathbf{i} \\ 4 & -2\mathbf{i} \end{pmatrix}, E_3 = \begin{pmatrix} -8\mathbf{i} & 17+36\mathbf{i} \\ -2-12\mathbf{i} & 4+34\mathbf{i} \end{pmatrix}.$$

Check that

$$r_{a_ie_i|0d_i} = r_{a_i} + r_{d_i} = \begin{cases} 4, & \text{if } i = 1, 3 \\ 3, & \text{if } i = 2 \end{cases},\ r_{b_i0|e_ic_i} = r_{b_i} + r_{c_i} = \begin{cases} 2, & \text{if } i = 1 \\ 3, & \text{if } i = 2 \\ 4, & \text{if } i = 3 \end{cases},$$

$$r_{a_ie_ic_i} = r_{a_ic_i} = 2,\ r_{a_je_jc_j00|0d_j0b_{j+1}0|00a_{j+1}-e_{j+1}c_{j+1}} = r_{a_jc_j0|0a_{j+1}c_{j+1}} + r_{d_jb_{j+1}} = 7,$$

$$r_{b_i|e_i|d_i} = r_{b_i|d_i} = 2,\ r_{b_j00|e_jc_j0|d_j0b_{j+1}|0a_{j+1}-e_{j+1}|00d_{j+1}} = r_{c_j|a_{j+1}} + r_{b_j0|d_jb_{j+1}|0d_{j+1}} = 7,$$

$$r_{a_je_jc_j0|0d_j0b_{j+1}|00a_{j+1}-e_{j+1}|000d_{j+1}} = r_{a_jc_j|0a_{j+1}} + r_{d_jb_{j+1}|0d_{j+1}} = \begin{cases} 8, & \text{if } j = 1 \\ 7, & \text{if } j = 2 \end{cases},$$

$$r_{b_j000|e_jc_j00|d_j0b_{j+1}0|0a_{j+1}-e_{j+1}c_{j+1}} = r_{c_j0|a_{j+1}c_{j+1}} + r_{b_j0|d_jb_{j+1}} = \begin{cases} 6, & \text{if } j = 1 \\ 8, & \text{if } j = 2 \end{cases},$$

$$\begin{aligned} &r_{a_1e_1c_10000|0d_10b_2000|00a_2-e_2c_200|000d_20b_30|0000a_3e_3c_3} \\ &= r_{a_1c_100|0a_2c_20|00a_3c_3} + r_{d_1b_20|0d_2b_3} = 12, \end{aligned}$$

$$\begin{aligned} &r_{b_10000|e_1c_1000|d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3|0000d_3} \\ &= r_{c_10|a_2c_2|0a_3} + r_{b_100|d_1b_20|0d_2b_3|00d_3} = 11, \end{aligned}$$

$$\begin{aligned} &r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3|00000d_3} \\ &= r_{a_1c_10|0a_2c_2|00a_3} + r_{d_1b_20|0d_2b_3|00d_3} = 12, \end{aligned}$$

$$\begin{aligned} &r_{b_100000|e_1c_10000|d_10b_2000|0a_2-e_2c_200|00d_20b_30|000a_3e_3c_3} \\ &= r_{c_100|a_2c_20|0a_3c_3} + r_{b_100|d_1b_20|0d_2b_3} = 11. \end{aligned}$$

All the equalities in (18)–(26) hold. Hence, the system of two-sided coupled generalized Sylvester matrix equations (1) is consistent.

Remark 3.4. Some special cases of Theorem 3.1 are also known; examples include the one-sided coupled generalized Sylvester matrix equations (6) [55], the one-sided generalized mixed Sylvester matrix equations (4) and (5) [29,37,54], the two-sided matrix equation $A_1XB_1 + C_1YD_1 = E_1$ [5,41,38,62], the system of two generalized Sylvester equations $A_iX_iB_i + C_iX_{i+1}D_i = E_i$ $(i = 1, 2)$, and the classical matrix equations (7) [41,42,60].

Remark 3.5. We can give the solvability conditions and general solutions to the system of two generalized Sylvester equations $A_iX_iB_i + C_iX_{i+1}D_i = E_i$ $(i = 1, 2)$, and one-sided coupled generalized Sylvester matrix equations (6) using the approach presented in [55]. Notice however, it is hard to give the solvability conditions and general solutions to the system (1) using the approach presented in [55]. This can reflect the advantages of our approach.

3.2. General solutions to the system of two-sided coupled generalized Sylvester matrix equations (1)

In this section, we derive the general solutions to the system (1) when it is solvable.

Theorem 3.6. *Let* $A_i \in \mathbb{C}^{p_i\times t_i}, B_i \in \mathbb{C}^{s_i\times q_i}, C_i \in \mathbb{C}^{p_i\times t_{i+1}}$, *and* $D_i \in \mathbb{C}^{s_{i+1}\times q_i}$ *be given,* $(i = 1, 2, 3)$. *Assume that the system of two-sided coupled generalized Sylvester matrix equations (1) has a solution* $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1}\times\mathbb{C}^{t_2\times s_2}\times\mathbb{C}^{t_3\times s_3}\times\mathbb{C}^{t_4\times s_4}$. *Then, the general solutions to the system (1) can be expressed as*

$$X_1 = T_1Y_1S_1, \quad X_2 = T_2Y_2S_2, \quad X_3 = T_3Y_3S_3, \quad X_4 = T_4Y_4S_4, \tag{46}$$

where

$$Y_1 = \begin{array}{c} \\ n_{1;1}\\ n_{1;2}\\ n_{1;3}\\ n_{1;4}\\ n_{1;5}\\ n_{1;6}\\ t_1-r_{a_1}\end{array}\begin{array}{c}\begin{array}{ccccccc} n_{2;1} & n_{2;2} & n_{2;3} & n_{2;4} & n_{2;5} & n_{2;6} & s_1-r_{b_1}\end{array}\\ \begin{pmatrix} E^1_{11}-E^2_{11}+E^3_{11}-Y^4_{11} & E^1_{12}-E^2_{12}+E^3_{12} & E^1_{13}-E^2_{13}+Y^3_{13} & E^1_{14}-E^2_{14} & E^1_{15}-Y^2_{15} & E^1_{16} & Y^1_{17}\\ E^1_{21}-E^2_{21}+E^3_{21} & E^1_{22}-E^2_{22}+E^3_{22} & E^1_{23}-E^2_{23}+Y^3_{23} & E^1_{24}-E^2_{24} & E^1_{25}-Y^2_{25} & E^1_{26} & Y^1_{27}\\ E^1_{31}-E^2_{31}+Y^3_{31} & E^1_{32}-E^2_{32}+Y^3_{32} & E^1_{33}-E^2_{33}+Y^3_{33} & E^1_{34}-E^2_{34} & E^1_{35}-Y^2_{35} & E^1_{36} & Y^1_{37}\\ E^1_{41}-E^2_{41} & E^1_{42}-E^2_{42} & E^1_{43}-E^2_{43} & E^1_{44}-E^2_{44} & E^1_{45}-Y^2_{45} & E^1_{46} & Y^1_{47}\\ E^1_{51}-Y^2_{51} & E^1_{52}-Y^2_{52} & E^1_{53}-Y^2_{53} & E^1_{54}-Y^2_{54} & E^1_{55}-Y^2_{55} & E^1_{56} & Y^1_{57}\\ E^1_{61} & E^1_{62} & E^1_{63} & E^1_{64} & E^1_{65} & E^1_{66} & Y^1_{67}\\ Y^1_{71} & Y^1_{72} & Y^1_{73} & Y^1_{74} & Y^1_{75} & Y^1_{76} & Y^1_{77}\end{pmatrix}\end{array}, \tag{47}$$

$$Y_2 = (Y^1_2,\ Y^2_2), \tag{48}$$

$$Y^1_2 = \begin{pmatrix} E^2_{11}-E^3_{11}+Y^4_{11} & E^2_{12}-E^3_{12} & E^2_{13}-Y^3_{13} & E^2_{14} & Y^2_{15} & E^1_{17} & E^1_{18} & E^1_{19}\\ E^2_{21}-E^3_{21} & E^2_{22}-E^3_{22} & E^2_{23}-Y^3_{23} & E^2_{24} & Y^2_{25} & E^1_{27} & E^1_{28} & E^1_{29}\\ E^2_{31}-Y^3_{31} & E^2_{32}-Y^3_{32} & E^2_{33}-Y^3_{33} & E^2_{34} & Y^2_{35} & E^1_{37} & E^1_{38} & E^1_{39}\\ E^2_{41} & E^2_{42} & E^2_{43} & E^2_{44} & Y^2_{45} & E^1_{47} & E^1_{48} & E^1_{49}\\ Y^2_{51} & Y^2_{52} & Y^2_{53} & Y^2_{54} & Y^2_{55} & E^1_{57} & E^1_{58} & E^1_{59}\\ E^1_{71} & E^1_{72} & E^1_{73} & E^1_{74} & E^1_{75} & E^1_{77} & E^1_{78} & E^1_{79}\\ E^1_{81} & E^1_{82} & E^1_{83} & E^1_{84} & E^1_{85} & E^1_{87} & E^1_{88} & E^1_{89}\\ E^1_{91} & E^1_{92} & E^1_{93} & E^1_{94} & E^1_{95} & E^1_{97} & E^1_{98} & E^1_{99}\\ E^1_{10,1} & E^1_{10,2} & E^1_{10,3} & E^1_{10,4} & E^1_{10,5} & E^1_{10,7} & E^1_{10,8} & E^1_{10,9}\\ E^1_{11,1} & E^1_{11,2} & E^1_{11,3} & E^1_{11,4} & E^1_{11,5} & E^1_{11,7} & E^1_{11,8} & E^1_{11,9}\\ E^2_{91}-E^3_{51}+Y^4_{31} & E^2_{92}-E^3_{52} & E^2_{93}-Y^3_{73} & E^2_{94} & Y^2_{11,5} & E^2_{95}-E^3_{53}+Y^4_{32} & E^2_{96}-E^3_{54} & E^2_{97}-Y^3_{76}\\ E^2_{10,1}-E^3_{61} & E^2_{10,2}-E^3_{62} & E^2_{10,3}-Y^3_{83} & E^2_{10,4} & Y^2_{12,5} & E^2_{10,5}-E^3_{63} & E^2_{10,6}-E^3_{64} & E^2_{10,7}-Y^3_{86}\\ E^2_{11,1}-Y^3_{91} & E^2_{11,2}-Y^3_{92} & E^2_{11,3}-Y^3_{93} & E^2_{11,4} & Y^2_{13,5} & E^2_{11,5}-Y^3_{94} & E^2_{11,6}-Y^3_{95} & E^2_{11,7}-Y^3_{96}\\ E^2_{12,1} & E^2_{12,2} & E^2_{12,3} & E^2_{12,4} & Y^2_{14,5} & E^2_{12,5} & E^2_{12,6} & E^2_{12,7}\\ Y^2_{15,1} & Y^2_{15,2} & Y^2_{15,3} & Y^2_{15,4} & Y^2_{15,5} & Y^2_{15,6} & Y^2_{15,7} & Y^2_{15,8}\end{pmatrix}, \tag{49}$$

$$Y_2^2 = \begin{pmatrix} E^1_{1,10} & E^1_{1,11} & E^2_{19}-E^3_{15}+Y^4_{13} & E^2_{1,10}-E^3_{16} & E^2_{1,11}-Y^3_{19} & E^2_{1,12} & Y^2_{1,15} \\ E^1_{2,10} & E^1_{2,11} & E^2_{29}-E^3_{25} & E^2_{2,10}-E^3_{26} & E^2_{2,11}-Y^3_{29} & E^2_{2,12} & Y^2_{2,15} \\ E^1_{3,10} & E^1_{3,11} & E^2_{39}-Y^3_{37} & E^2_{3,10}-Y^3_{38} & E^2_{3,11}-Y^3_{39} & E^2_{3,12} & Y^2_{3,15} \\ E^1_{4,10} & E^1_{4,11} & E^2_{49} & E^2_{4,10} & E^2_{4,11} & E^2_{4,12} & Y^2_{4,15} \\ E^1_{5,10} & E^1_{5,11} & Y^2_{5,11} & Y^2_{5,12} & Y^2_{5,13} & Y^2_{5,14} & Y^2_{5,15} \\ E^1_{7,10} & E^1_{7,11} & E^2_{59}-E^3_{35}+Y^4_{23} & E^2_{5,10}-E^3_{36} & E^2_{5,11}-Y^3_{49} & E^2_{5,12} & Y^2_{6,15} \\ E^1_{8,10} & E^1_{8,11} & E^2_{69}-E^3_{45} & E^2_{6,10}-E^3_{46} & E^2_{6,11}-Y^3_{59} & E^2_{6,12} & Y^2_{7,15} \\ E^1_{9,10} & E^1_{9,11} & E^2_{79}-Y^3_{67} & E^2_{7,10}-Y^3_{68} & E^2_{7,11}-Y^3_{69} & E^2_{7,12} & Y^2_{8,15} \\ E^1_{10,10} & E^1_{10,11} & E^2_{89} & E^2_{8,10} & E^2_{8,11} & E^2_{8,12} & Y^2_{9,15} \\ E^1_{11,10} & E^1_{11,11} & Y^2_{10,11} & Y^2_{10,12} & Y^2_{10,13} & Y^2_{10,14} & Y^2_{10,15} \\ E^2_{10,8} & Y^2_{12,10} & E^2_{10,9}-Y^3_{87} & E^2_{10,10}-Y^3_{88} & E^2_{10,11}-Y^3_{89} & E^2_{10,12} & Y^2_{12,15} \\ E^2_{11,8} & Y^2_{13,10} & E^2_{11,9}-Y^3_{97} & E^2_{11,10}-Y^3_{98} & E^2_{11,11}-Y^3_{99} & E^2_{11,12} & Y^2_{13,15} \\ E^2_{12,8} & E^2_{14,10} & E^2_{12,9} & E^2_{12,10} & E^2_{12,11} & E^2_{12,12} & Y^2_{14,15} \\ Y^2_{15,9} & Y^2_{15,10} & Y^2_{15,11} & Y^2_{15,12} & Y^2_{15,13} & Y^2_{15,14} & Y^2_{15,15} \end{pmatrix}, \tag{50}$$

$$Y_3 = \begin{pmatrix} E^3_{11}-Y^4_{11} & E^3_{12} & Y^3_{13} & E^2_{15}-E^1_{17} & E^2_{16}-E^1_{18} & E^2_{17}-E^1_{19} & E^3_{15}-Y^4_{13} & E^3_{16} & Y^3_{19} & E^2_{1,13} & E^2_{1,14} & E^2_{1,15} & E^3_{19}-Y^4_{15} & E^3_{1,10} & Y^3_{1,15} \\ E^3_{21} & E^3_{22} & Y^3_{23} & E^2_{25}-E^1_{27} & E^2_{26}-E^1_{28} & E^2_{27}-E^1_{29} & E^3_{25} & E^3_{26} & Y^3_{29} & E^2_{2,13} & E^2_{2,14} & E^2_{2,15} & E^3_{29} & E^3_{2,10} & Y^3_{2,15} \\ Y^3_{31} & Y^3_{32} & Y^3_{33} & E^2_{35}-E^1_{37} & E^2_{36}-E^1_{38} & E^2_{37}-E^1_{39} & Y^3_{37} & Y^3_{38} & Y^3_{39} & E^2_{3,13} & E^2_{3,14} & E^2_{3,15} & Y^3_{3,13} & Y^3_{3,14} & Y^3_{3,15} \\ E^2_{51}-E^1_{71} & E^2_{52}-E^1_{72} & E^2_{53}-E^1_{73} & E^2_{55}-E^1_{77} & E^2_{56}-E^1_{78} & E^2_{57}-E^1_{79} & E^3_{35}-Y^4_{23} & E^3_{36} & Y^3_{49} & E^2_{5,13} & E^2_{5,14} & E^2_{5,15} & E^3_{39}-Y^4_{25} & E^3_{3,10} & Y^3_{4,15} \\ E^2_{61}-E^1_{81} & E^2_{62}-E^1_{82} & E^2_{63}-E^1_{83} & E^2_{65}-E^1_{87} & E^2_{66}-E^1_{88} & E^2_{67}-E^1_{89} & E^3_{45} & E^3_{46} & Y^3_{59} & E^2_{6,13} & E^2_{6,14} & E^2_{6,15} & E^3_{49} & E^3_{4,10} & Y^3_{5,15} \\ E^2_{71}-E^1_{91} & E^2_{72}-E^1_{92} & E^2_{73}-E^1_{93} & E^2_{75}-E^1_{97} & E^2_{76}-E^1_{98} & E^2_{77}-E^1_{99} & Y^3_{67} & Y^3_{68} & Y^3_{69} & E^2_{7,13} & E^2_{7,14} & E^2_{7,15} & Y^3_{6,13} & Y^3_{6,14} & Y^3_{6,15} \\ E^3_{51}-Y^4_{31} & E^3_{52} & Y^3_{73} & E^3_{53}-Y^4_{32} & E^3_{54} & Y^3_{76} & E^3_{55}-Y^4_{33} & E^3_{56} & Y^3_{79} & E^2_{9,13} & E^2_{9,14} & E^2_{9,15} & E^3_{59}-Y^4_{35} & E^3_{5,10} & Y^3_{7,15} \\ E^3_{61} & E^3_{62} & Y^3_{83} & E^3_{63} & E^3_{64} & Y^3_{86} & E^3_{65} & E^3_{66} & Y^3_{89} & E^2_{10,13} & E^2_{10,14} & E^2_{10,15} & E^3_{69} & E^3_{6,10} & Y^3_{8,15} \\ Y^3_{91} & Y^3_{92} & Y^3_{93} & Y^3_{94} & Y^3_{95} & Y^3_{96} & Y^3_{97} & Y^3_{98} & Y^3_{99} & E^2_{11,13} & E^2_{11,14} & E^2_{11,15} & Y^3_{9,13} & Y^3_{9,14} & Y^3_{9,15} \\ E^2_{13,1} & E^2_{13,2} & E^2_{13,3} & E^2_{13,5} & E^2_{13,6} & E^2_{13,7} & E^2_{13,9} & E^2_{13,10} & E^2_{13,11} & E^2_{13,13} & E^2_{13,14} & E^2_{13,15} & E^3_{79}-Y^4_{45} & E^3_{7,10} & Y^3_{10,15} \\ E^2_{14,1} & E^2_{14,2} & E^2_{14,3} & E^2_{14,5} & E^2_{14,6} & E^2_{14,7} & E^2_{14,9} & E^2_{14,10} & E^2_{14,11} & E^2_{14,13} & E^2_{14,14} & E^2_{14,15} & E^3_{89} & E^3_{8,10} & Y^3_{11,15} \\ E^2_{15,1} & E^2_{15,2} & E^2_{15,3} & E^2_{15,5} & E^2_{15,6} & E^2_{15,7} & E^2_{15,9} & E^2_{15,10} & E^2_{15,11} & E^2_{15,13} & E^2_{15,14} & E^2_{15,15} & Y^3_{12,13} & Y^3_{12,14} & Y^3_{12,15} \\ E^3_{91}-Y^4_{51} & E^3_{92} & Y^3_{13,3} & E^3_{93}-Y^4_{52} & E^3_{94} & Y^3_{13,6} & E^3_{95}-Y^4_{53} & E^3_{96} & Y^3_{13,9} & E^3_{97}-Y^4_{54} & E^3_{98} & Y^3_{13,12} & E^3_{99}-Y^4_{55} & E^3_{9,10} & Y^3_{13,15} \\ E^3_{10,1} & E^3_{10,2} & Y^3_{14,3} & E^3_{10,3} & E^3_{10,4} & Y^3_{14,6} & E^3_{10,5} & E^3_{10,6} & Y^3_{14,9} & E^3_{10,7} & E^3_{10,8} & Y^3_{14,12} & E^3_{10,9} & E^3_{10,10} & Y^3_{14,15} \\ Y^3_{15,1} & Y^3_{15,2} & Y^3_{15,3} & Y^3_{15,4} & Y^3_{15,5} & Y^3_{15,6} & Y^3_{15,7} & Y^3_{15,8} & Y^3_{15,9} & Y^3_{15,10} & Y^3_{15,11} & Y^3_{15,12} & Y^3_{15,13} & Y^3_{15,14} & Y^3_{15,15} \end{pmatrix}, \tag{51}$$

$$Y_4 = \begin{array}{c} \\ n_{1;1} \\ n_{1;7} \\ n_{1;12} \\ n_{1;16} \\ n_{1;19} \\ n_{1;21} \\ t_4-r_{c_3} \end{array} \begin{array}{c} \begin{array}{ccccccc} n_{2;1} & n_{2;7} & n_{2;12} & n_{2;16} & n_{2;19} & n_{2;21} & s_4-r_{d_3} \end{array} \\ \begin{pmatrix} Y^4_{11} & E^3_{13}-E^2_{15}+E^1_{17} & Y^4_{13} & E^3_{17}-E^2_{1,13} & Y^4_{15} & E^3_{1,11} & Y^4_{17} \\ E^3_{31}-E^2_{51}+E^1_{71} & E^3_{33}-E^2_{55}+E^1_{77} & Y^4_{23} & E^3_{37}-E^2_{5,13} & Y^4_{25} & E^3_{3,11} & Y^4_{27} \\ Y^4_{31} & Y^4_{32} & Y^4_{33} & E^3_{57}-E^2_{9,13} & Y^4_{35} & E^3_{5,11} & Y^4_{37} \\ E^3_{71}-E^2_{13,1} & E^3_{73}-E^2_{13,5} & E^3_{75}-E^2_{13,9} & E^3_{77}-E^2_{13,13} & Y^4_{45} & E^3_{7,11} & Y^4_{47} \\ Y^4_{51} & Y^4_{52} & Y^4_{53} & Y^4_{54} & Y^4_{55} & E^3_{9,11} & Y^4_{57} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,5} & E^3_{11,7} & E^3_{11,9} & E^3_{11,11} & Y^4_{67} \\ Y^4_{71} & Y^4_{72} & Y^4_{73} & Y^4_{74} & Y^4_{75} & Y^4_{76} & Y^4_{77} \end{pmatrix} \end{array}, \tag{52}$$

where $T_1, T_2, T_3, T_4, S_1, S_2, S_3, S_4, E^1_{u_1v_1}, E^2_{u_2v_2}, E^3_{u_3v_3}$, *and* $E^4_{u_4v_4}$ *are defined in Theorem 2.2, and the remaining* $Y^1_{j_1k_1}$, $Y^2_{j_2k_2}, Y^3_{j_3k_3}$, *and* $Y^4_{j_4k_4}$ *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes.*

Proof. The expressions of the general solutions to the system (1) can be derived from (16) and (28)–(30). □

Remark 3.7. Let the dimensions of the arbitrary submatrices in Y_1, Y_2, Y_3, and Y_4 be equal to zero. Then we can give some necessary and sufficient conditions for a consistent system (1) to have a unique solution.

3.3. Range of ranks of X_1 *and* X_4 *in the system (1)*

Maximal and minimal ranks are found to be useful in control theory [12]. In this section, we consider the maximal and minimal ranks of X_1 and X_4 in the system (1) in terms of the ranks of the coefficient matrices.

Theorem 3.8. *Assume that the system of two-sided coupled generalized Sylvester matrix equations (1) has a solution* $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1}\times\mathbb{C}^{t_2\times s_2}\times\mathbb{C}^{t_3\times s_3}\times\mathbb{C}^{t_4\times s_4}$. *Denote*

$$S_1 = \left\{ X_4 \in \mathbb{C}^{t_4\times s_4} \;\middle|\; A_iX_iB_i + C_iX_{i+1}D_i = E_i \ (i = 1, 2, 3) \right\}.$$

Then,

$$\begin{aligned}\max_{X_4\in S_1} r(X_4) = \min\big\{ & t_4,\ s_4,\ t_4 + s_4 + r_{a_3e_3} - r_{a_3c_3} - r_{d_3}, t_4 + s_4 + r_{b_3|e_3} - r_{b_3|d_3} - r_{c_3},\\ & t_4 + s_4 + r_{a_2-e_2c_20|0d_20b_3|00a_3e_3} - r_{a_2c_20|0a_3c_3} - r_{d_2b_3|0d_3},\\ & t_4 + s_4 + r_{b_200|-e_2c_20|d_20b_3|0a_3e_3} - r_{c_20|a_3c_3} - r_{b_20|d_2b_3|0d_3},\\ & t_4 + s_4 + r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3}\\ & - r_{a_1c_100|0a_2c_20|00a_3c_3} - r_{d_1b_20|0d_2b_3|00d_3},\\ & t_4 + s_4 + r_{b_10000|e_1c_1000|d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3}\\ & - r_{c_100|a_2c_20|0a_3c_3} - r_{b_100|d_1b_20|0d_2b_3|00d_3}\big\},\end{aligned}$$

$$\begin{aligned}\min_{X_4\in S_1} r(X_4) = & r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3}\\ & + r_{b_10000|e_1c_1000|d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3}\\ & - r_{0b_10000|a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3}\\ & + r_{a_2-e_2c_20|0d_20b_3|00a_3e_3}\\ & - r_{d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3} + r_{b_200|-e_2c_20|d_20b_3|0a_3e_3}\end{aligned}$$

$$
\begin{aligned}
&- r_{c_1000|0b_200|a_2-e_2c_20|0d_20b_3|00a_3e_3} \\
&+ r_{b_3|e_3} - r_{c_20|0b_3|a_3e_3} + r_{a_3e_3} - r_{d_20b_3|0a_3e_3}.
\end{aligned}
$$

Proof. The proof is given in Appendix B. □

The range of ranks of X_1 in the system (1) are described in the following Theorem.

Theorem 3.9. *Assume that the system of two-sided coupled generalized Sylvester matrix equations (1) has a solution* $(X_1, X_2, X_3, X_4) \in \mathbb{C}^{t_1\times s_1}\times\mathbb{C}^{t_2\times s_2}\times\mathbb{C}^{t_3\times s_3}\times\mathbb{C}^{t_4\times s_4}$. *Denote*

$$
S_2 = \left\{ X_1 \in \mathbb{C}^{t_1\times s_1} \,\middle|\, A_iX_iB_i + C_iX_{i+1}D_i = E_i,\ (i = 1, 2, 3) \right\}.
$$

Then,

$$
\begin{aligned}
\max_{X_1\in S_2} r(X_1) = \min \big\{ & t_1,\ s_1,\ t_1 + s_1 + r_{c_1e_1} - r_{a_1c_1} - r_{b_1}, t_1 + s_1 + r_{d_1|e_1} - r_{b_1|d_1} - r_{a_1}, \\
& t_1 + s_1 + r_{c_2-e_2a_20|0b_20d_1|00c_1e_1} - r_{c_2a_20|0c_1a_1} - r_{b_2d_1|0b_1}, \\
& t_1 + s_1 + r_{d_200|-e_2a_20|b_20d_1|0c_1e_1} - r_{a_20|c_1a_1} - r_{d_20|b_2d_1|0b_1}, \\
& t_1 + s_1 + r_{c_3e_3a_3000|0b_30d_200|00c_2-e_2a_20|000b_20d_1|0000c_1e_1} \\
& - r_{c_3a_300|0c_2a_20|00c_1a_1} - r_{b_3d_20|0b_2d_1|00b_1}, \\
& t_1 + s_1 + r_{d_30000|e_3a_3000|b_30d_200|0c_2-e_2a_20|00b_20d_1|000c_1e_1} \\
& - r_{a_300|c_2a_20|0c_1a_1} - r_{d_300|b_3d_20|0b_2d_1|00b_1} \big\},
\end{aligned}
$$

$$
\begin{aligned}
&\min_{X_1\in S_2} r(X_1) \\
&= r_{c_3e_3a_3000|0b_30d_200|00c_2-e_2a_20|000b_20d_1|0000c_1e_1} \\
&+ r_{d_30000|e_3a_3000|b_30d_200|0c_2-e_2a_20|00b_20d_1|000c_1e_1} \\
&- r_{0d_30000|c_3e_3a_3000|0b_30d_200|00c_2-e_2a_20|000b_20d_1|0000c_1e_1} + r_{c_2-e_2a_20|0b_20d_1|00c_1e_1} \\
&- r_{b_30d_200|0c_2-e_2a_20|00b_20d_1|000c_1e_1} + r_{d_200|-e_2a_20|b_20d_1|0c_1e_1} \\
&- r_{a_3000|0d_200|c_2-e_2a_20|0b_20d_1|00c_1e_1} + r_{d_1|e_1} - r_{a_20|0d_1|c_1e_1} + r_{c_1e_1} - r_{b_20d_1|0c_1e_1}.
\end{aligned}
$$

Example 3.10. Consider the system of two-sided coupled generalized Sylvester matrix equations (1) with the given matrices in Example 3.3. Then,

$$
\max_{X_4\in S_1} r(X_4) = 3, \quad \min_{X_4\in S_1} r(X_4) = 0, \quad \min_{X_1\in S_2} r(X_1) = 3, \quad \min_{X_1\in S_2} r(X_1) = 1.
$$

Remark 3.11. Note that the one-sided coupled generalized Sylvester matrix equations (6) are special cases of the system (1). The authors in [55] do not only give the necessary and sufficient conditions for the existence of a solution, but they also provide the general

solutions to (6) using the Moore–Penrose inverses of the matrices. Notice, however, that it is hard to derive the maximal and minimal ranks of the general solutions to the first and third set of equations in (6) using the method presented in [55]. This is not the case for the approach proposed in this paper.

Remark 3.12. We may consider the simultaneous decomposition for k matrices with compatible dimensions over an arbitrary division ring with an involutive antiautomorphism $*$, denoted by $\mathcal{F}$. For $A_i \in \mathcal{F}^{n_{i-1}\times n_i}$ $(i = 1, 2, \ldots, k)$, we can find nonsingular matrices $X_i \in \mathcal{F}^{n_i\times n_i}$ such that

$$X_{i-1}^* A_i X_i = \Lambda_i, \tag{53}$$

where Λ_i are defined in Lemma 2.1. Based on (53), the similar results about the system (1) over $\mathcal{F}$ can be obtained.

4. The simultaneous decomposition for $5k$ matrices and the system of k two-sided coupled generalized Sylvester matrix equations

In this section, we consider the more general system, i.e., the system (2), where $A_i \in \mathbb{C}^{p_i\times t_i}$, $B_i \in \mathbb{C}^{s_i\times q_i}$, $C_i \in \mathbb{C}^{p_i\times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1}\times q_i}$, and $E_i \in \mathbb{C}^{p_i\times q_i}$ are given matrices. Note that the coefficient matrices A_i, B_i, C_i, D_i, and E_i can be arranged in the following matrix array

$$\begin{array}{c} \\ s_1 \\ p_1 \\ s_2 \\ p_2 \\ s_2 \\ \vdots \\ s_k \\ p_k \\ s_{k+1} \end{array} \begin{array}{c} \begin{array}{ccccccccc} t_1 & q_1 & t_2 & q_2 & t_3 & \ldots & t_k & q_k & t_{k+1} \end{array} \\ \begin{pmatrix} & B_1 & & & & & & & \\ A_1 & E_1 & C_1 & & & & & & \\ & D_1 & & B_2 & & & & & \\ & & A_2 & E_2 & C_2 & & & & \\ & & & D_2 & & \ddots & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & & & B_k & \\ & & & & & & A_k & E_k & C_k \\ & & & & & & & D_k & \end{pmatrix} \end{array}. \tag{54}$$

In order to consider the solvability conditions and general solutions to the system (2), we need to construct a simultaneous decomposition of $5k$ matrices (54). We want to find nonsingular matrices $P_i \in \mathbb{C}^{p_i\times p_i}$, $Q_i \in \mathbb{C}^{q_i\times q_i}$, $T_i \in \mathbb{C}^{t_i\times t_i}$, and $S_i \in \mathbb{C}^{s_i\times s_i}$, such that

$$P_i A_i T_i = \Delta_{2i-1},\ P_i C_i T_{i+1} = \Delta_{2i},\ S_i B_i Q_i = \Omega_{2i-1},\ S_{i+1} D_i Q_i = \Omega_{2i},$$

$$P_i E_i Q_i = \Phi_i,\ i = 1, 2, \ldots, k,\ k \geq 4, \tag{55}$$

where $\Delta_{2i-1}, \Delta_{2i}, \Omega_{2i-1}, \Omega_{2i}$ are quasi-diagonal matrices with the finest possible subdivision of matrices, and Φ_i have appropriate forms. Using the approach of the simultaneous decomposition of 15 matrices (8), we can establish the simultaneous decomposition of $5k$ matrices (54) through the following three steps.

1. Applying Lemma 2.1 ($Q^{2k-1} - SVD$) to the matrices $A_1^*, C_1, A_2^*, C_2, \cdots, A_k^*$, and C_k, we can derive an equivalence canonical form of the set of $2k$ matrices:

$$\begin{pmatrix} A_1 & C_1 & & & \\ & A_2 & C_2 & & \\ & & \ddots & \ddots & \\ & & & A_k & C_k \end{pmatrix}. \tag{56}$$

   The equivalence canonical form of (56) is quasi-diagonal with the finest possible subdivision of matrices.
2. Similarly, we can derive an equivalence canonical form of the set of B_i and D_i:

$$\begin{pmatrix} B_1 & & & \\ D_1 & B_2 & & \\ & D_2 & \ddots & \\ & & \ddots & B_k \\ & & & D_k \end{pmatrix}. \tag{57}$$

   The equivalence canonical form of (57) is quasi-diagonal with the finest possible subdivision of matrices.
3. Applying elementary matrix operations and the identity matrices in the equivalence canonical forms of matrices (56) and (57), we can obtain a simultaneous decomposition of $5k$ matrices (54).
4. Applying the main results in [20], we can obtain all the dimensions of identity matrices in the structure.

Based on the simultaneous decomposition of $5k$ matrices (54) in (55), we can rewrite the system (2) in the following canonical form

$$P_i^{-1}\Delta_{2i-1}(T_i^{-1}X_iS_i^{-1})\Omega_{2i-1}Q_i^{-1} + P_i^{-1}\Delta_{2i}(T_{i+1}^{-1}X_{i+1}S_{i+1}^{-1})\Omega_{2i}Q_i^{-1} = P_i^{-1}\Phi_iQ_i^{-1}.$$

If we then let $Y_i = T_i^{-1}X_iS_i^{-1}$, the original system (2) transforms to

$$\Delta_{2i-1}Y_i\Omega_{2i-1} + \Delta_{2i}Y_{i+1}\Omega_{2i} = \Phi_i,\ i = 1, 2, \ldots, k,\ k \geq 4. \tag{58}$$

Now we have the following result.

Theorem 4.1. *The system of k two-sided coupled generalized Sylvester matrix equations (2) is consistent if and if the system of matrix equations (58) is consistent, where $\Delta_{2i-1}, \Delta_{2i}, \Omega_{2i-1}, \Omega_{2i}$, and Φ_i are given in (55). In this case, the general solutions X_i to system (2) can be expressed as $X_i = T_i Y_i S_i$, where T_i and S_i are given in (55).*

A more challenging problem is to give the solvability condition to the system of k two-sided coupled generalized Sylvester matrix equations (2) using rank conditions. Motivated by Theorem 3.1, we give a conjecture on the solvability condition to the system (2).

Conjecture 4.2. *The system of k two-sided generalized Sylvester matrix equations (2) is consistent if and only if the following $2k(k+1)$ rank equalities hold*

$$r_{a_i e_i c_i} = r_{a_i c_i}, r_{b_i|e_i|d_i} = r_{b_i|d_i}, r_{a_i e_i|0 d_i} = r_{a_i} + r_{d_i}, r_{b_i 0|e_i c_i} = r_{b_i} + r_{c_i}, i = 1, \ldots, k,$$

$$\theta^1_{m;n} = \alpha^1_{m;n} + \beta^1_{m;n}, \theta^2_{m;n} = \alpha^2_{m;n} + \beta^2_{m;n}, \theta^3_{m;n} = \alpha^3_{m;n} + \beta^3_{m;n},$$
$$\theta^4_{m;n} = \alpha^4_{m;n} + \beta^4_{m;n}, 1 \le m < n \le k,$$

where

$$\theta^1_{m;n} = r \begin{pmatrix} A_m & E_m & C_m & & & & \\ & D_m & & B_{m+1} & & & \\ & & A_{m+1} & -E_{m+1} & C_{m+1} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_n & (-1)^{n-m} E_n & C_n \end{pmatrix},$$

$$\theta^2_{m;n} = r \begin{pmatrix} B_m & & & & & \\ E_m & C_m & & & & \\ D_m & & B_{m+1} & & & \\ & A_{m+1} & -E_{m+1} & \ddots & & \\ & & D_{m+1} & \ddots & B_n & \\ & & & \ddots & (-1)^{n-m} E_n & \\ & & & & D_n & \end{pmatrix},$$

$$\theta^3_{m;n} = r \begin{pmatrix} A_m & E_m & C_m & & & \\ & D_m & & B_{m+1} & & \\ & & A_{m+1} & -E_{m+1} & \ddots & \\ & & & D_{m+1} & \ddots & B_n \\ & & & & \ddots & (-1)^{n-m} E_n \\ & & & & & D_n \end{pmatrix},$$

$$\theta^4_{m;n} = r\begin{pmatrix} B_m & & & & & & \\ E_m & C_m & & & & & \\ D_m & & B_{m+1} & & & & \\ & A_{m+1} & -E_{m+1} & C_{m+1} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_n & (-1)^{n-m}E_n & C_n \end{pmatrix},$$

$$\alpha^1_{m;n} = r\begin{pmatrix} A_m & C_m & & & \\ & A_{m+1} & C_{m+1} & & \\ & & \ddots & \ddots & \\ & & & A_n & C_n \end{pmatrix},$$

$$\beta^1_{m;n} = r\begin{pmatrix} D_m & B_{m+1} & & & \\ & D_{m+1} & B_{m+2} & & \\ & & \ddots & \ddots & \\ & & & D_{n-1} & B_n \end{pmatrix},$$

$$\alpha^2_{m;n} = r\begin{pmatrix} C_m & & & \\ A_{m+1} & C_{m+1} & & \\ & A_{m+2} & \ddots & \\ & & \ddots & C_{n-1} \\ & & & A_n \end{pmatrix}, \beta^2_{m;n} = r\begin{pmatrix} B_m & & & \\ D_m & B_{m+1} & & \\ & D_{m+1} & \ddots & \\ & & \ddots & B_n \\ & & & D_n \end{pmatrix},$$

$$\alpha^3_{m;n} = r\begin{pmatrix} A_m & C_m & & & \\ & A_{m+1} & C_{m+1} & & \\ & & \ddots & \ddots & \\ & & & A_{n-1} & C_{n-1} \\ & & & & A_n \end{pmatrix},$$

$$\beta^3_{m;n} = r\begin{pmatrix} D_m & B_{m+1} & & & \\ & D_{m+1} & B_{m+2} & & \\ & & \ddots & \ddots & \\ & & & D_{n-1} & B_n \\ & & & & D_n \end{pmatrix},$$

$$\alpha^4_{m;n} = r\begin{pmatrix} C_m & & & \\ A_{m+1} & C_{m+1} & & \\ & A_{m+2} & C_{m+2} & \\ & & \ddots & \ddots \\ & & & A_n & C_n \end{pmatrix},$$

$$\beta^4_{m;n} = r\begin{pmatrix} B_m & & & \\ D_{m+1} & B_{m+1} & & \\ & D_{m+2} & B_{m+2} & \\ & & \ddots & \ddots \\ & & & D_{n-1} & B_n \end{pmatrix}.$$

Remark 4.3. Theorem 3.1 follows from letting A_i, B_i, C_i, D_i, E_i $(i = 4, \cdots, k)$ vanish in Conjecture 4.2. According to the method of solving the system (1), we can also prove this conjecture when k is a small integer though it is quite complicated. So far, we have not found a general method to prove the sufficiency of the conjecture.

5. Conclusion

We have derived a simultaneous decomposition for a set of fifteen matrices with compatible sizes: $A_i \in \mathbb{C}^{p_i \times t_i}$, $B_i \in \mathbb{C}^{s_i \times q_i}$, $C_i \in \mathbb{C}^{p_i \times t_{i+1}}$, $D_i \in \mathbb{C}^{s_{i+1} \times q_i}$, and $E_i \in \mathbb{C}^{p_i \times q_i}$ $(i = 1, 2, 3)$. We also have presented implicit expressions of the dimensions of identity matrices in the structure. Using this simultaneous decomposition, we have provided some necessary and sufficient conditions for the existence and the general solutions to the system of two-sided coupled generalized Sylvester matrix equations (1). We have given the range of ranks of X_1 and X_4 in the system (1). Moreover, we have presented a similar approach to consider the simultaneous decomposition for $5k$ matrices and the system of k two-sided coupled generalized Sylvester matrix equations with $k + 1$ unknowns (2). Some known results can be viewed as special cases of the ones obtained in this paper. It is worthy to say that the main results of this paper are also valid over the real number field and the real quaternion algebra. Moreover, we may establish a similar simultaneous decomposition for k matrices over an arbitrary division ring with an involutive antiautomorphism.

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Appendix A

We prove Theorem 2.2 according to the idea in §2.1.

Proof. We establish the result through the following three steps. First, we give an equivalence canonical form of the set of 6 matrices (10). Second, we derive an equivalence canonical form of the set of 6 matrices (11). Finally, we provide a simultaneous decomposition of the matrix array (8).

Step 1. Our first step is to give an equivalence canonical form of the set of 6 matrices (10). A $Q^5 - SVD$ (i.e., $k = 6$ in Lemma 2.1) of six matrices A_i^* and C_i takes the form

$$U_1^* A_1^* X_1 = \Lambda_1, X_1^* C_1 X_2 = \Lambda_2, X_2^* A_2^* X_3 = \Lambda_3,$$
$$X_3^* C_2 X_4 = \Lambda_4, X_4^* A_3^* X_5 = \Lambda_5, X_5^* C_3 X_6 = \Sigma_6, \tag{59}$$

where U_1, V_6 are unitary matrices, $X_1, X_2, X_3, X_4, X_5, X_6 = V_6 \mathrm{diag}(\Lambda_6^1, \Lambda_6^2, \Lambda_6^3, \Lambda_6^4, \Lambda_6^5, \Lambda_6^6)^{-1}$ are nonsingular matrices, and

$$\Lambda_1 = \begin{array}{c} \\ r_1^1 \\ t_1 - r_{a_1} \end{array} \begin{array}{c} \begin{array}{cc} r_1^1 & p_1 - r_{a_1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \Lambda_2 = \begin{array}{c} \\ r_2^1 \\ r_1^1 - r_2^1 \\ r_2^2 \\ p_1 - r_{a_1} - r_2^2 \end{array} \begin{array}{c} \begin{array}{ccc} r_2^1 & r_2^2 & t_2 - r_{c_1} \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$\Lambda_3 = \begin{array}{c} \\ r_3^1 \\ r_2^1 - r_3^1 \\ r_3^2 \\ r_2^2 - r_3^2 \\ r_3^3 \\ t_2 - r_{c_1} - r_3^3 \end{array} \begin{array}{c} \begin{array}{cccc} r_3^1 & r_3^2 & r_3^3 & p_2 - r_{a_2} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$\Lambda_4 = \begin{array}{c} \\ r_4^1 \\ r_3^1 - r_4^1 \\ r_4^2 \\ r_3^2 - r_4^2 \\ r_4^3 \\ r_3^3 - r_4^3 \\ r_4^4 \\ p_2 - r_{a_2} - r_4^4 \end{array} \begin{array}{c} \begin{array}{ccccc} r_4^1 & r_4^2 & r_4^3 & r_4^4 & t_3 - r_{c_2} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$\Lambda_5 = \begin{array}{c} \\ r_5^1 \\ r_4^1 - r_5^1 \\ r_5^2 \\ r_4^2 - r_5^2 \\ r_5^3 \\ r_4^3 - r_5^3 \\ r_5^4 \\ r_4^4 - r_5^4 \\ r_5^5 \\ t_3 - r_{c_2} - r_5^5 \end{array} \begin{array}{c} \begin{array}{cccccc} r_5^1 & r_5^2 & r_5^3 & r_5^4 & r_5^5 & p_3 - r_{a_3} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$\Sigma_6 = \begin{array}{c} \\ r_6^1 \\ r_5^1 - r_6^1 \\ r_6^2 \\ r_5^2 - r_6^2 \\ r_6^3 \\ r_5^3 - r_6^3 \\ r_6^4 \\ r_5^4 - r_6^4 \\ r_6^5 \\ r_5^5 - r_6^5 \\ r_6^6 \\ p_3 - r_{a_3} - r_6^6 \end{array} \begin{array}{c} \begin{array}{ccccccc} r_6^1 & r_6^2 & r_6^3 & r_6^4 & r_6^5 & r_6^6 & t_4 - r_{c_3} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Observe that

$$\operatorname{diag}(X_1^*, X_3^*, X_5^*) \begin{pmatrix} A_1 & C_1 & & \\ & A_2 & C_2 & \\ & & A_3 & C_3 \end{pmatrix} \operatorname{diag}(U_1, X_2, X_4, X_6) = \begin{pmatrix} \Lambda_1^* & \Lambda_2 & & \\ & \Lambda_3^* & \Lambda_4 & \\ & & \Lambda_5^* & \Sigma_6 \end{pmatrix}.$$

The matrices $\operatorname{diag}(X_1^*, X_3^*, X_5^*)$ and $\operatorname{diag}(U_1, X_2, X_4, X_6)$ are nonsingular. Hence, we can obtain expressions for all dimensions involved by analyzing the block bidiagonal matrix

$$\left(\begin{array}{cccc} \Lambda_1^* & \Lambda_2 & & \\ \hline & \Lambda_3^* & \Lambda_4 & \\ \hline & & \Lambda_5^* & \Sigma_6 \end{array}\right)$$

$$= \begin{array}{c} \\ n_{1;1} \\ n_{1;2} \\ n_{1;3} \\ n_{1;4} \\ n_{1;5} \\ n_{1;6} \\ n_{1;7} \\ n_{1;8} \\ n_{1;9} \\ n_{1;10} \\ n_{1;11} \\ p_1-r_{a_1c_1} \\ \hline n_{1;1} \\ n_{1;2} \\ n_{1;3} \\ n_{1;4} \\ n_{1;7} \\ n_{1;8} \\ n_{1;9} \\ n_{1;10} \\ n_{1;12} \\ n_{1;13} \\ n_{1;14} \\ n_{1;15} \\ n_{1;16} \\ n_{1;17} \\ n_{1;18} \\ p_2-r_{a_2c_2} \\ \hline n_{1;1} \\ n_{1;2} \\ n_{1;7} \\ n_{1;8} \\ n_{1;12} \\ n_{1;13} \\ n_{1;16} \\ n_{1;17} \\ n_{1;19} \\ n_{1;20} \\ n_{1;21} \\ p_3-r_{a_3c_3} \end{array} \left(\begin{array}{cccc} I\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,I\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,I\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,I\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,I\,0\,0 & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,I\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & & \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & & \\ \hline I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0 & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0 & \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0 & \\ \hline & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0 \\ & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0 \\ & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0 \\ & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,I\,0\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0 \\ & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \end{array}\right),$$

where we have used the finest possible subdivision of matrices, and

$$n_{1;1} := r_6^1, n_{1;2} := r_5^1 - r_6^1, n_{1;3} := r_4^1 - r_5^1, n_{1;4} := r_3^1 - r_4^1,$$
$$n_{1;5} := r_2^1 - r_3^1, n_{1;6} := r_1^1 - r_2^1,$$

$$n_{1;7} := r_6^2, n_{1;8} := r_5^2 - r_6^2, n_{1;9} := r_4^2 - r_5^2, n_{1;10} := r_3^2 - r_4^2, n_{1;11} := r_2^2 - r_3^2,$$

$$n_{1;12} := r_6^3, n_{1;13} := r_5^3 - r_6^3, n_{1;14} := r_4^3 - r_5^3, n_{1;15} := r_3^3 - r_4^3,$$

$$n_{1;16} := r_6^4, n_{1;17} := r_5^4 - r_6^4, n_{1;18} := r_4^4 - r_5^4, n_{1;19} := r_6^5, n_{1;20} := r_5^5 - r_6^5, n_{1;21} := r_6^6.$$

Expressions for the integers $n_{1;1}, \ldots, n_{1;21}$ are given in Theorem 2.4.

Step 2. Next consider a similar decomposition of 6 matrices (11). Applying the approach in Step 1 to the conjugate transpose of (11), we have

$$\operatorname{diag}(Y_1^*, Y_3^*, Y_5^*) \begin{pmatrix} B_1^* & D_1^* & & \\ & B_2^* & D_2^* & \\ & & B_3^* & D_3^* \end{pmatrix} \operatorname{diag}(U_2, Y_2, Y_4, Y_6) = \begin{pmatrix} \Theta_1^* & \Theta_2 & & \\ & \Theta_3^* & \Theta_4 & \\ & & \Theta_5^* & \Theta_6 \end{pmatrix},$$

where U_2 is unitary, $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ are nonsingular matrices, and

$$\begin{pmatrix} \Theta_1^* & \Theta_2 & & \\ & \Theta_3^* & \Theta_4 & \\ & & \Theta_5^* & \Theta_6 \end{pmatrix}$$

$$= \begin{array}{c|cccc}
 & n_{2;1} & n_{2;1} & n_{2;1} & \\
 & n_{2;2} & n_{2;2} & n_{2;12} & \\
 & n_{2;3} & n_{2;3} & n_{2;13} & \\
 & n_{2;4} & n_{2;4} & n_{2;16} & \\
 & n_{2;5} & n_{2;7} & n_{2;17} & \\
 & n_{2;6} & n_{2;8} & n_{2;19} & \\
 & n_{2;7} & n_{2;9} & n_{2;20} & \\
 & n_{2;8} & n_{2;10} & n_{2;21} & \\
 & n_{2;9} & n_{2;11} & q_3 - r_{b_3|d_3} & \\
 & n_{2;10} & q_1 - r_{b_1|d_1} & & \\
 & n_{2;11} & & & \\
q_1 - r_{b_1|d_1} & & & & \\
\hline
n_{2;1} & I\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;2} & 0\,I\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;3} & 0\,0\,I\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;4} & 0\,0\,0\,I\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;5} & 0\,0\,0\,0\,I\,0\,0 & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;6} & 0\,0\,0\,0\,0\,I\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;7} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;8} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;9} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;10} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;11} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & \\
q_1 - r_{b_1|d_1} & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
\hline
n_{2;1} & & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;2} & & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;3} & & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;4} & & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;7} & & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;8} & & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;9} & & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;10} & & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;12} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;13} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;14} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & \\
n_{2;15} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
n_{2;16} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & \\
n_{2;17} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & \\
n_{2;18} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0 & \\
q_2 - r_{b_2|d_2} & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\
\hline
n_{2;1} & & & I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & I\,0\,0\,0\,0\,0\,0 \\
n_{2;2} & & & 0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\
n_{2;7} & & & 0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,I\,0\,0\,0\,0\,0 \\
n_{2;8} & & & 0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\
n_{2;12} & & & 0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,I\,0\,0\,0\,0 \\
n_{2;13} & & & 0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\
n_{2;16} & & & 0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,I\,0\,0\,0 \\
n_{2;17} & & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\
n_{2;19} & & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,I\,0\,0 \\
n_{2;20} & & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\
n_{2;21} & & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,I\,0\,0\,0 & 0\,0\,0\,0\,0\,I\,0 \\
q_3 - r_{b_3|d_3} & & & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0
\end{array}.$$

Expressions for the integers $n_{2;1}, \ldots, n_{2;21}$ are given in Theorem 2.4.

Step 3. Now we turn our attention to the matrix array (8), i.e., the set of 15 matrices:

$$A_i \in \mathbb{C}^{p_i \times t_i},\ B_i \in \mathbb{C}^{s_i \times q_i},\ C_i \in \mathbb{C}^{p_i \times t_{i+1}},\ D_i \in \mathbb{C}^{s_{i+1} \times q_i},\ E_i \in \mathbb{C}^{p_i \times q_i}\ (i = 1, 2, 3).$$

By Step 1 and Step 2, we can find two unitary matrices U_1 and U_2, nonsingular matrices $X_1, \ldots, X_6,\ Y_1, \ldots, Y_6$ such that

$$X_1^* A_1 U_1 = \Lambda_1^*,\ X_1^* C_1 X_2 = \Lambda_2,\ X_3^* A_2 X_2 = \Lambda_3^*,\ X_3^* C_2 X_4 = \Lambda_4,$$
$$X_5^* A_3 X_4 = \Lambda_5^*,\ X_5^* C_3 X_6 = \Sigma_6,$$

$$U_2^* B_1 Y_1 = \Theta_1,\ Y_2^* D_1 Y_1 = \Theta_2^*,\ Y_2^* B_2 Y_3 = \Theta_3,\ Y_4^* D_2 Y_3 = \Theta_4^*,$$
$$Y_4^* B_3 Y_5 = \Theta_5,\ Y_6^* D_3 Y_5 = \Theta_6^*.$$

Let us partition the matrices $X_1^* E_1 Y_1$, $X_2^* E_2 Y_2$, and $X_3^* E_3 Y_3$ into

$$X_1^* E_1 Y_1 =: \left((E_{uv}^1)^{(1)}\right)_{12\times 12}, X_3^* E_2 Y_3 =: \left((E_{uv}^2)^{(1)}\right)_{16\times 16}, X_5^* E_3 Y_5 =: \left((E_{uv}^3)^{(1)}\right)_{12\times 12}.$$

The block rows of $\left((E_{uv}^1)^{(1)}\right)_{12\times 12}$, $\left((E_{uv}^2)^{(1)}\right)_{16\times 16}$, and $\left((E_{uv}^3)^{(1)}\right)_{12\times 12}$ are the same as the block rows of Λ_1^*, Λ_3^*, and Λ_5^*, respectively. The block columns of $\left((E_{uv}^1)^{(1)}\right)_{12\times 12}$, $\left((E_{uv}^2)^{(1)}\right)_{16\times 16}$, and $\left((E_{uv}^3)^{(1)}\right)_{12\times 12}$ are the same as the block rows of Θ_1^*, Θ_3^*, and Θ_5^*, respectively. For the matrices $(E_{12,12}^1)^{(1)}$, $(E_{16,16}^2)^{(1)}$, and $(E_{12,12}^3)^{(1)}$, there exist nonsingular matrices X_7, Y_7, X_8, Y_8, X_9, and Y_9 such that

$$X_7^* (E_{12,12}^1)^{(1)} Y_7 = \begin{pmatrix} 0 & 0 \\ 0 & I_{l_1} \end{pmatrix},\ X_8^* (E_{16,16}^2)^{(1)} Y_8 = \begin{pmatrix} 0 & 0 \\ 0 & I_{l_2} \end{pmatrix},$$
$$X_9^* (E_{12,12}^3)^{(1)} Y_9 = \begin{pmatrix} 0 & 0 \\ 0 & I_{l_3} \end{pmatrix},$$

where $l_1 := r\left((E_{12,12}^1)^{(1)}\right)$, $l_2 := r\left((E_{16,16}^2)^{(1)}\right)$, $l_3 := r\left((E_{12,12}^3)^{(1)}\right)$. Let

$$X_7^* \left((E_{12,1}^1)^{(1)}, \cdots, (E_{12,11}^1)^{(1)}\right) =: \begin{pmatrix} (E_{12,1}^1)^{(2)} & \cdots & (E_{12,11}^1)^{(2)} \\ (E_{13,1}^1)^{(2)} & \cdots & (E_{13,11}^1)^{(2)} \end{pmatrix},$$
$$\begin{pmatrix} (E_{1,12}^1)^{(1)} \\ \vdots \\ (E_{11,12}^1)^{(1)} \end{pmatrix} Y_7 =: \begin{pmatrix} (E_{1,12}^1)^{(2)} & (E_{1,13}^1)^{(2)} \\ \vdots & \vdots \\ (E_{11,12}^1)^{(2)} & (E_{11,13}^1)^{(2)} \end{pmatrix},$$

$$X_8^* \left((E_{16,1}^2)^{(1)}, \cdots, (E_{16,15}^2)^{(1)}\right) =: \begin{pmatrix} (E_{16,1}^2)^{(2)} & \cdots & (E_{16,15}^2)^{(2)} \\ (E_{17,1}^2)^{(2)} & \cdots & (E_{17,15}^2)^{(2)} \end{pmatrix},$$
$$\begin{pmatrix} (E_{1,16}^2)^{(1)} \\ \vdots \\ (E_{15,16}^2)^{(1)} \end{pmatrix} Y_8 =: \begin{pmatrix} (E_{1,16}^2)^{(2)} & (E_{1,17}^2)^{(2)} \\ \vdots & \vdots \\ (E_{15,16}^2)^{(2)} & (E_{15,17}^2)^{(2)} \end{pmatrix},$$

$$X_9^*\left((E_{12,1}^3)^{(1)},\cdots,(E_{12,11}^3)^{(1)}\right)=:\begin{pmatrix}(E_{12,1}^3)^{(2)} & \cdots & (E_{12,11}^3)^{(2)}\\ (E_{13,1}^3)^{(2)} & \cdots & (E_{13,11}^3)^{(2)}\end{pmatrix},$$

$$\begin{pmatrix}(E_{1,12}^3)^{(1)}\\ \vdots\\ (E_{11,12}^3)^{(1)}\end{pmatrix}Y_9=:\begin{pmatrix}(E_{1,12}^3)^{(2)} & (E_{1,13}^3)^{(2)}\\ \vdots & \vdots\\ (E_{11,12}^3)^{(2)} & (E_{11,13}^3)^{(2)}\end{pmatrix}.$$

Then, we have

$$\begin{aligned}&\begin{pmatrix}I_{\gamma_1} & 0\\ 0 & X_7^*\end{pmatrix}X_1^*E_1Y_1\begin{pmatrix}I_{\tau_1} & 0\\ 0 & Y_7\end{pmatrix}\\ &=:\begin{pmatrix}(E_{11}^1)^{(2)} & \cdots & (E_{1,11}^1)^{(2)} & (E_{1,12}^1)^{(2)} & (E_{1,13}^1)^{(2)}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ (E_{11,1}^1)^{(2)} & \cdots & (E_{11,11}^1)^{(2)} & (E_{11,12}^1)^{(2)} & (E_{11,13}^1)^{(2)}\\ (E_{12,1}^1)^{(2)} & \cdots & (E_{12,11}^1)^{(2)} & 0 & 0\\ (E_{13,1}^1)^{(2)} & \cdots & (E_{13,11}^1)^{(2)} & 0 & I_{l_1}\end{pmatrix},\end{aligned}\tag{60}$$

$$\begin{aligned}&\begin{pmatrix}I_{\gamma_2} & 0\\ 0 & X_8^*\end{pmatrix}X_3^*E_2Y_3\begin{pmatrix}I_{\tau_2} & 0\\ 0 & Y_8\end{pmatrix}\\ &=:\begin{pmatrix}(E_{11}^2)^{(2)} & \cdots & (E_{1,15}^2)^{(2)} & (E_{1,16}^2)^{(2)} & (E_{1,17}^2)^{(2)}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ (E_{15,1}^2)^{(2)} & \cdots & (E_{15,15}^2)^{(2)} & (E_{15,16}^2)^{(2)} & (E_{15,17}^2)^{(2)}\\ (E_{16,1}^2)^{(2)} & \cdots & (E_{16,15}^2)^{(2)} & 0 & 0\\ (E_{17,1}^2)^{(2)} & \cdots & (E_{17,15}^2)^{(2)} & 0 & I_{l_2}\end{pmatrix},\end{aligned}\tag{61}$$

$$\begin{aligned}&\begin{pmatrix}I_{\gamma_3} & 0\\ 0 & X_9^*\end{pmatrix}X_5^*E_3Y_5\begin{pmatrix}I_{\tau_3} & 0\\ 0 & Y_9\end{pmatrix}\\ &=:\begin{pmatrix}(E_{11}^3)^{(2)} & \cdots & (E_{1,11}^3)^{(2)} & (E_{1,12}^3)^{(2)} & (E_{1,13}^3)^{(2)}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ (E_{11,1}^3)^{(2)} & \cdots & (E_{11,11}^3)^{(2)} & (E_{11,12}^3)^{(2)} & (E_{11,13}^3)^{(2)}\\ (E_{12,1}^3)^{(2)} & \cdots & (E_{12,11}^3)^{(2)} & 0 & 0\\ (E_{13,1}^3)^{(2)} & \cdots & (E_{13,11}^3)^{(2)} & 0 & I_{l_3}\end{pmatrix},\end{aligned}\tag{62}$$

where $\gamma_i=r_{a_ic_i}$, $\tau_i=r_{b_i|d_i}$, $(i=1,2,3)$. Let

$$\varphi_w:=\begin{pmatrix}(E_{1,13}^w)^{(2)}\\ \vdots\\ (E_{11,13}^w)^{(2)}\end{pmatrix},\ \phi_w:=\begin{pmatrix}(E_{13,1}^w)^{(2)}, & \cdots, & (E_{13,11}^w)^{(2)}\end{pmatrix},\ w=1,3,$$

$$\varphi_2 := \begin{pmatrix} (E^2_{1,17})^{(2)} \\ \vdots \\ (E^2_{15,17})^{(2)} \end{pmatrix},\ \phi_2 := \left((E^2_{17,1})^{(2)},\ \cdots,\ (E^2_{17,15})^{(2)} \right),$$

$$M_i := \begin{pmatrix} I_{\gamma_i} & 0 & -\varphi_i \\ 0 & I_{p_i-\gamma_i-l_i} & 0 \\ 0 & 0 & I_{l_i} \end{pmatrix},\ N_i := \begin{pmatrix} I_{\tau_i} & 0 & 0 \\ 0 & I_{s_i-\tau_i-l_i} & 0 \\ -\phi_i & 0 & I_{l_i} \end{pmatrix}.$$

Using I_{l_1}, I_{l_2}, and I_{l_3} as the pivots to eliminate the last block rows and columns of the right matrices in (60)–(62), we obtain

$$M_1 \begin{pmatrix} I_{\gamma_1} & 0 \\ 0 & X_7^* \end{pmatrix} X_1^* E_1 Y_1 \begin{pmatrix} I_{\tau_1} & 0 \\ 0 & Y_7 \end{pmatrix} N_1 =: \Phi_1,$$

$$M_2 \begin{pmatrix} I_{\gamma_2} & 0 \\ 0 & X_8^* \end{pmatrix} X_3^* E_2 Y_3 \begin{pmatrix} I_{\tau_2} & 0 \\ 0 & Y_8 \end{pmatrix} N_2 =: \Phi_2,$$

$$M_3 \begin{pmatrix} I_{\gamma_3} & 0 \\ 0 & X_9^* \end{pmatrix} X_5^* E_3 Y_5 \begin{pmatrix} I_{\tau_3} & 0 \\ 0 & Y_9 \end{pmatrix} N_3 =: \Phi_3,$$

where Φ_i are defined in (14). Upon computation, we obtain

$$M_1 \begin{pmatrix} I_{\gamma_1} & 0 \\ 0 & X_7^* \end{pmatrix} X_1^* A_1 U_1 =: \Delta_1, M_1 \begin{pmatrix} I_{\gamma_1} & 0 \\ 0 & X_7^* \end{pmatrix} X_1^* C_1 X_2 =: \Delta_2,$$
$$M_2 \begin{pmatrix} I_{\gamma_2} & 0 \\ 0 & X_8^* \end{pmatrix} X_3^* A_2 X_2 =: \Delta_3, M_2 \begin{pmatrix} I_{\gamma_2} & 0 \\ 0 & X_8^* \end{pmatrix} X_3^* C_2 X_4 =: \Delta_4,$$

$$M_3 \begin{pmatrix} I_{\gamma_3} & 0 \\ 0 & X_9^* \end{pmatrix} X_5^* A_3 X_4 =: \Delta_5, M_3 \begin{pmatrix} I_{\gamma_3} & 0 \\ 0 & X_9^* \end{pmatrix} X_5^* C_3 X_6 =: \Delta_6,$$
$$U_2^* B_1 Y_1 \begin{pmatrix} I_{\tau_1} & 0 \\ 0 & Y_7 \end{pmatrix} N_1 =: \Omega_1, Y_2^* D_1 Y_1 \begin{pmatrix} I_{\tau_1} & 0 \\ 0 & Y_7 \end{pmatrix} N_1 =: \Omega_2,$$

$$Y_2^* B_2 Y_3 \begin{pmatrix} I_{\tau_2} & 0 \\ 0 & Y_8 \end{pmatrix} N_2 =: \Omega_3, Y_4^* D_2 Y_3 \begin{pmatrix} I_{\tau_2} & 0 \\ 0 & Y_8 \end{pmatrix} N_2 =: \Omega_4,$$
$$Y_4^* B_3 Y_5 \begin{pmatrix} I_{\tau_3} & 0 \\ 0 & Y_9 \end{pmatrix} N_3 =: \Omega_5, Y_6^* D_3 Y_5 \begin{pmatrix} I_{\tau_3} & 0 \\ 0 & Y_9 \end{pmatrix} N_3 =: \Omega_6,$$

where $\Delta_1, \ldots, \Delta_6, \Omega_1, \ldots, \Omega_6$ are defined in (12) and (13). Let

$$P_1 := M_1 \begin{pmatrix} I_{\gamma_1} & 0 \\ 0 & X_7^* \end{pmatrix} X_1^*,\ P_2 := M_2 \begin{pmatrix} I_{\gamma_2} & 0 \\ 0 & X_8^* \end{pmatrix} X_3^*,\ P_3 := M_3 \begin{pmatrix} I_{\gamma_3} & 0 \\ 0 & X_9^* \end{pmatrix} X_5^*, \tag{63}$$

$$Q_1 := Y_1 \begin{pmatrix} I_{\tau_1} & 0 \\ 0 & Y_7 \end{pmatrix} N_1,\ Q_2 := Y_3 \begin{pmatrix} I_{\tau_2} & 0 \\ 0 & Y_8 \end{pmatrix} N_2,\ Q_3 := Y_5 \begin{pmatrix} I_{\tau_3} & 0 \\ 0 & Y_9 \end{pmatrix} N_3, \quad (64)$$

$$T_1 := U_1^*,\ T_2 := X_2,\ T_3 := X_4,\ T_4 := X_6,\ S_1 := U_2,\ S_2 := Y_2^*,\ S_3 := Y_4^*,\ S_4 := Y_6^*. \quad (65)$$

Hence, the unitary matrices $T_1 \in \mathbb{C}^{t_1\times t_1}$, $S_1 \in \mathbb{C}^{s_1\times s_1}$ and the nonsingular matrices $P_i \in \mathbb{C}^{p_i\times p_i}$, $Q_i \in \mathbb{C}^{q_i\times q_i}$, $T_2 \in \mathbb{C}^{t_2\times t_2}$, $T_3 \in \mathbb{C}^{t_3\times t_3}$, $T_4 \in \mathbb{C}^{t_4\times t_4}$, $S_2 \in \mathbb{C}^{s_2\times s_2}$, $S_3 \in \mathbb{C}^{s_3\times s_3}$, and $S_4 \in \mathbb{C}^{s_4\times s_4}$ in (63)–(65) satisfy the equation (15). □

Appendix B

Now we prove Theorem 3.8. We begin with the following lemmas.

Lemma B.1. *(See [13,15,61].) Let* $M_2(X,Y) = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$, *where* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{p\times q}$ *are given, and* $X \in \mathbb{C}^{m\times q}$ *and* $Y \in \mathbb{C}^{p\times n}$ *are variable matrices. Then,*

$$\max_{X,Y} r\left[M_2(X,Y)\right] = \min\left\{m+p, n+q, r_a+p+q, r_b+m+n\right\},$$

$$\min_{X,Y} r\left[M_2(X,Y)\right] = \max\left\{r_a,\ r_b\right\}.$$

Lemma B.2. *(See [15,61].) Let* $M_3 = \begin{pmatrix} A & B \\ D & Y \end{pmatrix}$, *where* A, B *and* D *are given, and* $Y \in \mathbb{C}^{n\times m}$ *is a variable matrix. Then,*

$$\max_{Y\in\mathbb{C}^{n\times m}} r\left(M_3\right) = \min\left\{n + r_{ab}, m + r_{a|d}\right\},\ \min_{Y\in\mathbb{C}^{n\times m}} r\left(M_3\right) = r_{ab} + r_{a|d} - r_a.$$

Now we are ready to present a proof for Theorem 3.8.

Proof. It follows from Theorem 3.6 that the expression of X_4 can be expressed as $X_4 = T_4Y_4S_4$, where Y_4 are given by (52). Clearly, $\underset{X_4\in S_1}{r}(X_4) = r(T_4Y_4S_4) = r(Y_4)$. Now we consider the range of ranks of Y_4. The main idea of this proof is to repeatedly use Lemma B.1 and Lemma B.2 to Y_4. We establish the result through the following three steps.

Step 1. Applying Lemma B.1 to the variable matrices $(Y_{71}^4, Y_{72}^4, Y_{73}^4, Y_{74}^4, Y_{75}^4, Y_{76}^4)$ and $\begin{pmatrix} Y_{17}^4 \\ Y_{27}^4 \\ Y_{37}^4 \\ Y_{47}^4 \\ Y_{57}^4 \\ Y_{67}^4 \end{pmatrix}$ of Y_4 gives

$$\max_{Y_{k7}^4, Y_{7k}^4, k=1,\dots,6} r\left(Y_4\right) = \min\left\{t_4,\ s_4,\ r(\Psi_1) + t_4 + s_4 - r_{c_3} - r_{d_3},\ r(Y_{77}^4) + r_{c_3} + r_{d_3}\right\},$$

$$\min_{Y_{k7}^4, Y_{7k}^4, k=1,\dots,6} r\left(Y_4\right) = \max\left\{r(\Psi_1),\quad r(Y_{77}^4)\right\},$$

where

$$\Psi_1 = \begin{array}{c} \\ n_{1;1} \\ n_{1;7} \\ n_{1;12} \\ n_{1;16} \\ n_{1;19} \\ n_{1;21} \end{array} \begin{array}{c} \begin{array}{cccccc} n_{2;1} & n_{2;7} & n_{2;12} & n_{2;16} & n_{2;19} & n_{2;21} \end{array} \\ \begin{pmatrix} Y_{11}^4 & E_{13}^3 - E_{15}^2 + E_{17}^1 & Y_{13}^4 & E_{17}^3 - E_{1,13}^2 & Y_{15}^4 & E_{1,11}^3 \\ E_{31}^3 - E_{51}^2 + E_{71}^1 & E_{33}^3 - E_{55}^2 + E_{77}^1 & Y_{23}^4 & E_{37}^3 - E_{5,13}^2 & Y_{25}^4 & E_{3,11}^3 \\ Y_{31}^4 & Y_{32}^4 & Y_{33}^4 & E_{57}^3 - E_{9,13}^2 & Y_{35}^4 & E_{5,11}^3 \\ E_{71}^3 - E_{13,1}^2 & E_{73}^3 - E_{13,5}^2 & E_{75}^3 - E_{13,9}^2 & E_{77}^3 - E_{13,13}^2 & Y_{45}^4 & E_{7,11}^3 \\ Y_{51}^4 & Y_{52}^4 & Y_{53}^4 & Y_{54}^4 & Y_{55}^4 & E_{9,11}^3 \\ E_{11,1}^3 & E_{11,3}^3 & E_{11,5}^3 & E_{11,7}^3 & E_{11,9}^3 & E_{11,11}^3 \end{pmatrix} \end{array}.$$

Observe that $\max r\left(Y_{77}^4\right) = \min\left\{t_4 - r_{c_3},\ s_4 - r_{d_3}\right\},\ \min r\left(Y_{77}^4\right) = 0$. Hence, we obtain

$$\max_{\substack{Y_{k7}^4, Y_{7k}^4, k=1,\dots,6, \\ Y_{77}^4}} r\left(Y_4\right) = \min\left\{t_4, s_4, r(\Psi_1) + t_4 + s_4 - r_{c_3} - r_{d_3}\right\},$$

$$\min_{\substack{Y_{k7}^4, Y_{7k}^4, k=1,\dots,6, \\ Y_{77}^4}} r\left(Y_4\right) = r(\Psi_1).$$

Step 2. Applying Lemma B.2 to the variable matrices $(Y_{51}^4, Y_{52}^4, Y_{53}^4, Y_{54}^4, Y_{55}^4)$ of Ψ_1 gives

$$\begin{aligned} &\max_{(Y_{51}^4, Y_{52}^4, Y_{53}^4, Y_{54}^4, Y_{55}^4)} r\left(\Psi_1\right) \\ &\quad = \min\left\{ n_{1;19} + r(\Psi_2), n_{2;1} + n_{2;7} + n_{2;12} + n_{2;16} + n_{2;19} + r\begin{pmatrix} E_{1,11}^3 \\ E_{3,11}^3 \\ E_{5,11}^3 \\ E_{7,11}^3 \\ E_{11,11}^3 \\ E_{9,11}^3 \end{pmatrix} \right\} \\ &\quad =: f(Y), \end{aligned}$$

$$\min_{(Y_{51}^4, Y_{52}^4, Y_{53}^4, Y_{54}^4, Y_{55}^4)} r\left(\Psi_1\right) = r(\Psi_2) + r\begin{pmatrix} E_{1,11}^3 \\ E_{3,11}^3 \\ E_{5,11}^3 \\ E_{7,11}^3 \\ E_{11,11}^3 \\ E_{9,11}^3 \end{pmatrix} - r\begin{pmatrix} E_{1,11}^3 \\ E_{3,11}^3 \\ E_{5,11}^3 \\ E_{7,11}^3 \\ E_{11,11}^3 \end{pmatrix} =: g(Y),$$

where

$$\Psi_2 = \begin{array}{c} \\ n_{1;1} \\ n_{1;7} \\ n_{1;12} \\ n_{1;16} \\ n_{1;21} \end{array} \begin{array}{c} \begin{array}{cccccc} n_{2;1} & n_{2;7} & n_{2;12} & n_{2;16} & n_{2;19} & n_{2;21} \end{array} \\ \begin{pmatrix} Y_{11}^4 & E_{13}^3 - E_{15}^2 + E_{17}^1 & Y_{13}^4 & E_{17}^3 - E_{1,13}^2 & Y_{15}^4 & E_{1,11}^3 \\ E_{31}^3 - E_{51}^2 + E_{71}^1 & E_{33}^3 - E_{55}^2 + E_{77}^1 & Y_{23}^4 & E_{37}^3 - E_{5,13}^2 & Y_{25}^4 & E_{3,11}^3 \\ Y_{31}^4 & Y_{32}^4 & Y_{33}^4 & E_{57}^3 - E_{9,13}^2 & Y_{35}^4 & E_{5,11}^3 \\ E_{71}^3 - E_{13,1}^2 & E_{73}^3 - E_{13,5}^2 & E_{75}^3 - E_{13,9}^2 & E_{77}^3 - E_{13,13}^2 & Y_{45}^4 & E_{7,11}^3 \\ E_{11,1}^3 & E_{11,3}^3 & E_{11,5}^3 & E_{11,7}^3 & E_{11,9}^3 & E_{11,11}^3 \end{pmatrix} \end{array}.$$

Using the similar approach of Step 2 to Ψ_2, we can obtain the maximal and minimal ranks of Y_4

$$\begin{aligned} &\max r\left(Y_4\right) \\ &= \left(\max_{Y_{11}^4}\left(\max_{(Y_{13}^4, Y_{23}^4)^*}\left(\max_{(Y_{31}^4, Y_{32}^4, Y_{33}^4)}\left(\max_{(Y_{15}^4, Y_{25}^4, Y_{35}^4, Y_{45}^4)^*} f(Y)\right)\right)\right)\right) \qquad (66) \\ &= \min\left\{ t_4, s_4, t_4 + s_4 - r_{c_3} - r_{d_3} + n_{2;1} + n_{2;7} + n_{2;12} + n_{2;16} + n_{2;19} + r\begin{pmatrix} E_{1,11}^3 \\ E_{3,11}^3 \\ E_{5,11}^3 \\ E_{7,11}^3 \\ E_{11,11}^3 \\ E_{9,11}^3 \end{pmatrix}, \right. \\ &\qquad \left. \begin{array}{l} t_4 + s_4 - r_{c_3} - r_{d_3} + n_{1;19} + n_{1;1} + n_{1;7} + n_{1;12} + n_{1;16} \\ \quad + r(E_{11,1}^3, E_{11,3}^3, E_{11,5}^3, E_{11,7}^3, E_{11,11}^3, E_{11,9}^3), \end{array} \right. \qquad (67) \\ &\qquad \left. t_4 + s_4 - r_{c_3} - r_{d_3} + n_{1;19} + n_{2;19} + n_{2;1} + n_{2;7} + n_{2;12} + r\begin{pmatrix} E_{17}^3 - E_{1,13}^2 & E_{1,11}^3 \\ E_{37}^3 - E_{5,13}^2 & E_{3,11}^3 \\ E_{77}^3 - E_{13,13}^2 & E_{7,11}^3 \\ E_{11,7}^3 & E_{11,11}^3 \\ E_{57}^3 - E_{9,13}^2 & E_{5,11}^3 \end{pmatrix}, \right. \\ &\qquad \left. \begin{array}{l} t_4 + s_4 - r_{c_3} - r_{d_3} + n_{1;19} + n_{2;19} + n_{1;12} + n_{1;1} + n_{1;7} \\ \quad + r\begin{pmatrix} E_{71}^3 - E_{13,1}^2 & E_{73}^3 - E_{13,5}^2 & E_{77}^3 - E_{13,13}^2 & E_{7,11}^3 & E_{75}^3 - E_{13,9}^2 \\ E_{11,1}^3 & E_{11,3}^3 & E_{11,7}^3 & E_{11,11}^3 & E_{11,5}^3 \end{pmatrix}, \\ t_4 + s_4 - r_{c_3} - r_{d_3} + n_{1;19} + n_{2;19} + n_{1;12} + n_{2;12} + n_{1;1} \\ \quad + r\begin{pmatrix} E_{31}^3 - E_{51}^2 + E_{71}^1 & E_{33}^3 - E_{55}^2 + E_{77}^1 & E_{37}^3 - E_{5,13}^2 & E_{3,11}^3 \\ E_{71}^3 - E_{13,1}^2 & E_{73}^3 - E_{13,5}^2 & E_{77}^3 - E_{13,13}^2 & E_{7,11}^3 \\ E_{11,1}^3 & E_{11,3}^3 & E_{11,7}^3 & E_{11,11}^3 \end{pmatrix}, \end{array} \right. \end{aligned}$$

$$t_4 + s_4 - r_{c_3} - r_{d_3} + n_{1;19} + n_{2;19} + n_{1;12} + n_{2;12} + n_{2;1}$$
$$+ r \left\{ \begin{pmatrix} E^3_{13} - E^2_{15} + E^1_{17} & E^3_{17} - E^2_{1,13} & E^3_{1,11} \\ E^3_{33} - E^2_{55} + E^1_{77} & E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix} \right\}, \tag{69}$$

$$\begin{aligned} \min r\left(Y_4\right) &= \left(\min_{Y^4_{11}} \left(\min_{(Y^4_{13}, Y^4_{23})^*} \left(\min_{(Y^4_{31}, Y^4_{32}, Y^4_{33})} \left(\min_{(Y^4_{15}, Y^4_{25}, Y^4_{35}, Y^4_{45})^*} g(Y) \right) \right) \right) \right) \\ &= r \begin{pmatrix} E^3_{31} - E^2_{51} + E^1_{71} & E^3_{33} - E^2_{55} + E^1_{77} & E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{71} - E^2_{13,1} & E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix} \\ &\quad + r \begin{pmatrix} E^3_{13} - E^2_{15} + E^1_{17} & E^3_{17} - E^2_{1,13} & E^3_{1,11} \\ E^3_{33} - E^2_{55} + E^1_{77} & E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix} \\ &\quad - r \begin{pmatrix} E^3_{33} - E^2_{55} + E^1_{77} & E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix} \\ &\quad + r \begin{pmatrix} E^3_{71} - E^2_{13,1} & E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} & E^3_{75} - E^2_{13,9} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} & E^3_{11,5} \end{pmatrix} \\ &\quad - r \begin{pmatrix} E^3_{71} - E^2_{13,1} & E^3_{73} - E^2_{13,5} & E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix} \\ &\quad + r \begin{pmatrix} E^3_{17} - E^2_{1,13} & E^3_{1,11} \\ E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,7} & E^3_{11,11} \\ E^3_{57} - E^2_{9,13} & E^3_{5,11} \end{pmatrix} - r \begin{pmatrix} E^3_{17} - E^2_{1,13} & E^3_{1,11} \\ E^3_{37} - E^2_{5,13} & E^3_{3,11} \\ E^3_{77} - E^2_{13,13} & E^3_{7,11} \\ E^3_{11,7} & E^3_{11,11} \end{pmatrix} \\ &\quad + r \begin{pmatrix} E^3_{1,11} \\ E^3_{3,11} \\ E^3_{5,11} \\ E^3_{7,11} \\ E^3_{11,11} \\ E^3_{9,11} \end{pmatrix} - r \begin{pmatrix} E^3_{1,11} \\ E^3_{3,11} \\ E^3_{5,11} \\ E^3_{7,11} \\ E^3_{11,11} \end{pmatrix} \\ &\quad + r(E^3_{11,1}, E^3_{11,3}, E^3_{11,5}, E^3_{11,7}, E^3_{11,11}, E^3_{11,9}) \\ &\quad - r(E^3_{11,1}, E^3_{11,3}, E^3_{11,5}, E^3_{11,7}, E^3_{11,11}). \end{aligned} \tag{70}$$

Step 3. Now we want to give the ranks of the block matrices in (69)–(70) using A_i, B_i, C_i, D_i, E_i, $n_{j;1}, \ldots, n_{j;21}$, and the notation of the rank of the block matrix (9). Note that the equalities in (31)–(45) hold. Upon computation, we obtain

$$r_{a_3e_3} = r_{a_3} + r(E^3_{11,1}, E^3_{11,3}, E^3_{11,5}, E^3_{11,7}, E^3_{11,11}, E^3_{11,9}),$$
$$r_{d_20b_3|0a_3e_3} = r_{d_2b_3} + r_{a_3} + r(E^3_{11,1}, E^3_{11,3}, E^3_{11,5}, E^3_{11,7}, E^3_{11,11}),$$

$$r_{b_3|e_3} = r_{b_3} + r\begin{pmatrix} E^3_{1,11} \\ E^3_{3,11} \\ E^3_{5,11} \\ E^3_{7,11} \\ E^3_{11,11} \\ E^3_{9,11} \end{pmatrix}, \quad r_{c_20|0b_3|a_3e_3} = r_{c_2|a_3} + r_{b_3} + r\begin{pmatrix} E^3_{1,11} \\ E^3_{3,11} \\ E^3_{5,11} \\ E^3_{7,11} \\ E^3_{11,11} \end{pmatrix},$$

$$r_{b_200|-e_2c_20|d_20b_3|0a_3e_3} = r_{c_2|a_3} + r_{b_20|d_2b_3} + r\begin{pmatrix} E^3_{17}-E^2_{1,13} & E^3_{1,11} \\ E^3_{37}-E^2_{5,13} & E^3_{3,11} \\ E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,7} & E^3_{11,11} \\ E^3_{57}-E^2_{9,13} & E^3_{5,11} \end{pmatrix},$$

$$r_{c_1000|0b_200|a_2-e_2c_20|0d_20b_3|00a_3e_3} = r_{c_10|a_2c_2|0a_3} + r_{b_20|d_2b_3} + r\begin{pmatrix} E^3_{17}-E^2_{1,13} & E^3_{1,11} \\ E^3_{37}-E^2_{5,13} & E^3_{3,11} \\ E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,7} & E^3_{11,11} \end{pmatrix},$$

$$r_{d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3}$$
$$= r_{d_1b_20|0d_2b_3} + r_{a_2c_2|0a_3} + r\begin{pmatrix} E^3_{71}-E^2_{13,1} & E^3_{73}-E^2_{13,5} & E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix},$$

$$r_{a_2-e_2c_20|0d_20b_3|00a_3e_3}$$
$$= r_{a_2c_2|0a_3} + r_{d_2b_3} + r\begin{pmatrix} E^3_{71}-E^2_{13,1} & E^3_{73}-E^2_{13,5} & E^3_{77}-E^2_{13,13} & E^3_{7,11} & E^3_{75}-E^2_{13,9} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} & E^3_{11,5} \end{pmatrix},$$

$$r_{b_10000|e_1c_1000|d_10b_200|0a_2-e_2c_20|00d_20b_3|000a_3e_3}$$
$$= r_{c_10|a_2c_2|0a_3} + r_{b_100|d_1b_20|0d_2b_3} + r\begin{pmatrix} E^3_{13}-E^2_{15}+E^1_{17} & E^3_{17}-E^2_{1,13} & E^3_{1,11} \\ E^3_{33}-E^2_{55}+E^1_{77} & E^3_{37}-E^2_{5,13} & E^3_{3,11} \\ E^3_{73}-E^2_{13,5} & E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix},$$

$$r_{a_1e_1c_1000|0d_10b_200|00a_2-e_2c_20|000d_20b_3|0000a_3e_3}$$
$$= r_{a_1c_10|0a_2c_2|00a_3} + r_{d_1b_20|0d_2b_3} + r\begin{pmatrix} E^3_{31}-E^2_{51}+E^1_{71} & E^3_{33}-E^2_{55}+E^1_{77} & E^3_{37}-E^2_{5,13} & E^3_{3,11} \\ E^3_{71}-E^2_{13,1} & E^3_{73}-E^2_{13,5} & E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,1} & E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix},$$

$$r_{0b_1 0000|a_1e_1c_1000|0d_1 0b_2 00|00a_2-e_2c_2 0|000d_2 0b_3|0000a_3e_3}$$
$$= r_{a_1c_1 0|0a_2c_2|00a_3} + r_{b_1 00|d_1b_2 0|0d_2b_3} + r\begin{pmatrix} E^3_{33}-E^2_{55}+E^1_{77} & E^3_{37}-E^2_{5,13} & E^3_{3,11} \\ E^3_{73}-E^2_{13,5} & E^3_{77}-E^2_{13,13} & E^3_{7,11} \\ E^3_{11,3} & E^3_{11,7} & E^3_{11,11} \end{pmatrix}.$$

Hence, we have proved Theorem 3.8. □

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